Worksheet 7: Some exercises (prep for lecture).

April 17, 2020

1 Introduction

Today I am going to give a few short exercises based on the lectures so far, in preparation for today's lecture.

2 Conformal mappings between domains in the plane

Recall that a conformal mapping is a mapping of sets $\Omega \to \Omega'$ which is a continuous, real differentiable bijection and takes angles between curves to angles between curves. (A conformal *map* is a continuous mapping which takes angle to angles, but drops the bijection requirement). We have access to the following results.

Proposition 1. *1. If $f : \Omega \to \Omega'$ is a conformal mapping then the compositional inverse mapping, f^{-1} (as a mapping of sets), is also conformal.*

2. If $f : \Omega \to \Omega'$ and $g : \Omega' \to \Omega''$ are conformal then $g \circ f : \Omega \to \Omega''$ is also conformal.

Theorem 1. *If Ω, Ω' are two open domains in the complex plane $\mathbb{C}$ then $f : \Omega \to \Omega'$ is a conformal mapping if and only if f is holomorphic and a bijection.*

Now conformal mappings give us a way to compare complex domains in a new way.

Question 1. *Construct a conformal mapping from the (open) unit disk D_1 to the disk D_2 of radius 2.*

Question 2. *Let $H = \{z \mid \mathrm{Im}(z) > 0$ be the upper half-plane. Show that for any fractional linear function $f = \frac{az+b}{cz+d}$ such that $a, b, c, d \in \mathbb{R}$ are* real *parameters* and the determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is positive *(thanks to Ankit for pointing this out), f determines a conformal mapping from H to H.*

Question 3. *Construct a conformal mapping from the (open) upper half-plane, $H = \{z \mid \mathrm{Im}(z) > 0$ to the (open) unit disk, D_1. Hint: try using a fractional linear transformation (but not with real coefficients in this case).*

Question 4. *(a) Construct a conformal mapping* $f : S^{vert}_{(-\pi,\pi)} \to \Omega$ *from the* vertical *complex strip* $S^{vert}_{(-\pi,\pi)}$ *given by* $\{z \mid -\pi < \mathrm{Re}(z) < \pi\}$ *and the complement to the ray of negative numbers,* $\Omega := \mathbb{C} \setminus \{-r \mid r \in \mathbb{R}_{\geq 0}$. *(Note:* Ω *is open, so does not contain* 0*).*
(b) What is the inverse $f^{-1} : \Omega \to S$*? Check that, indeed, this is a holomorphic function with nowhere zero derivative.*

Question 5. *Construct a conformal mapping* $f : S^{vert}_{(0,\pi)} \to H$ *from the vertical complex strip of numbers of real part between* 0 *and* π *to the upper half-plane* H.