

Worksheet 7: Some exercises (prep for lecture).

April 17, 2020

1 Introduction

Today I am going to give a few short exercises based on the lectures so far, in preparation for today's lecture.

2 Conformal mappings between domains in the plane

Recall that a conformal mapping is a mapping of sets $\Omega \rightarrow \Omega'$ which is a continuous, real differentiable bijection and takes angles between curves to angles between curves. (A conformal *map* is a continuous mapping which takes angle to angles, but drops the bijection requirement). We have access to the following results.

Proposition 1. 1. If $f : \Omega \rightarrow \Omega'$ is a conformal mapping then the compositional inverse mapping, f^{-1} (as a mapping of sets), is also conformal.

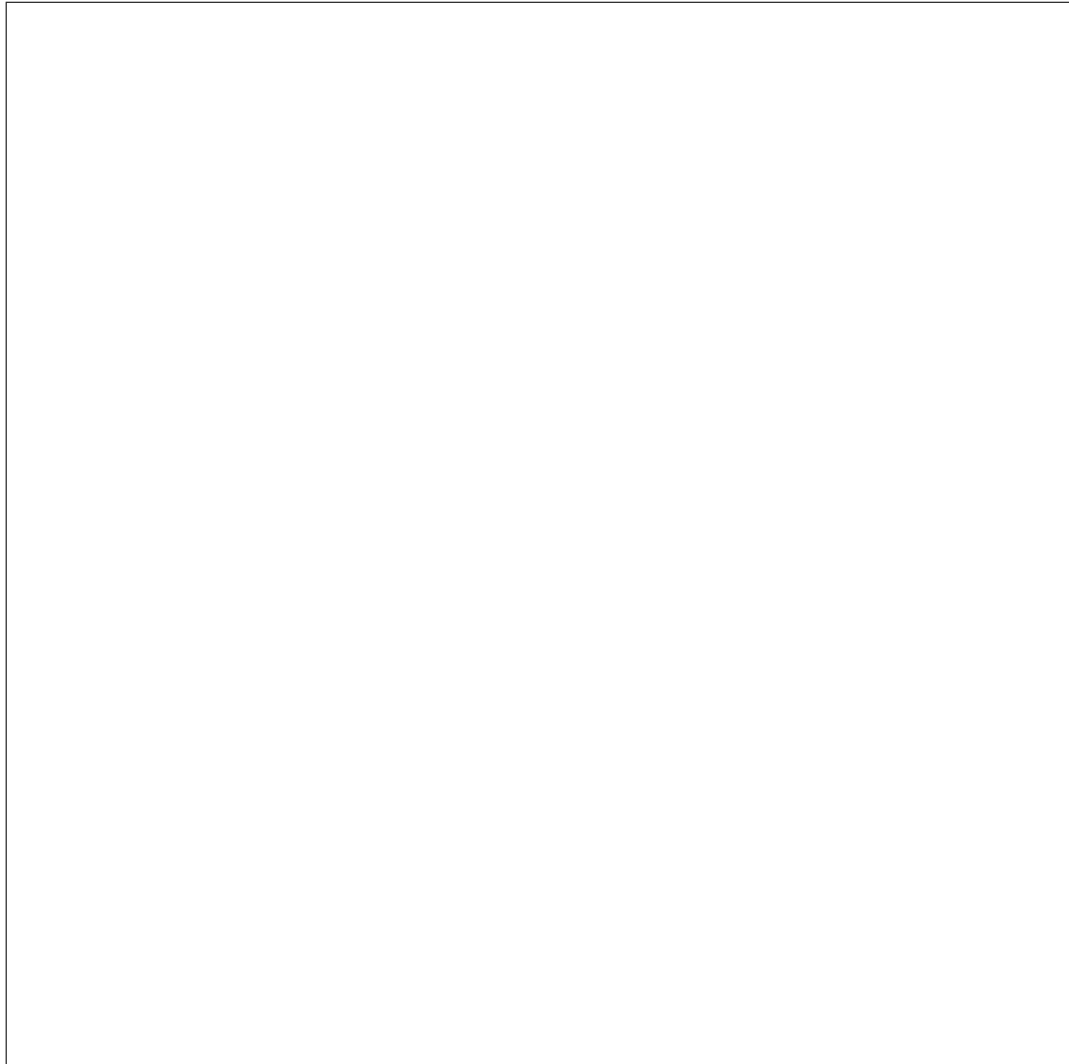
2. If $f : \Omega \rightarrow \Omega'$ and $g : \Omega' \rightarrow \Omega''$ are conformal then $g \circ f : \Omega \rightarrow \Omega''$ is also conformal.

Theorem 1. If Ω, Ω' are two open domains in the complex plane \mathbb{C} then $f : \Omega \rightarrow \Omega'$ is a conformal mapping if and only if f is holomorphic and a bijection.

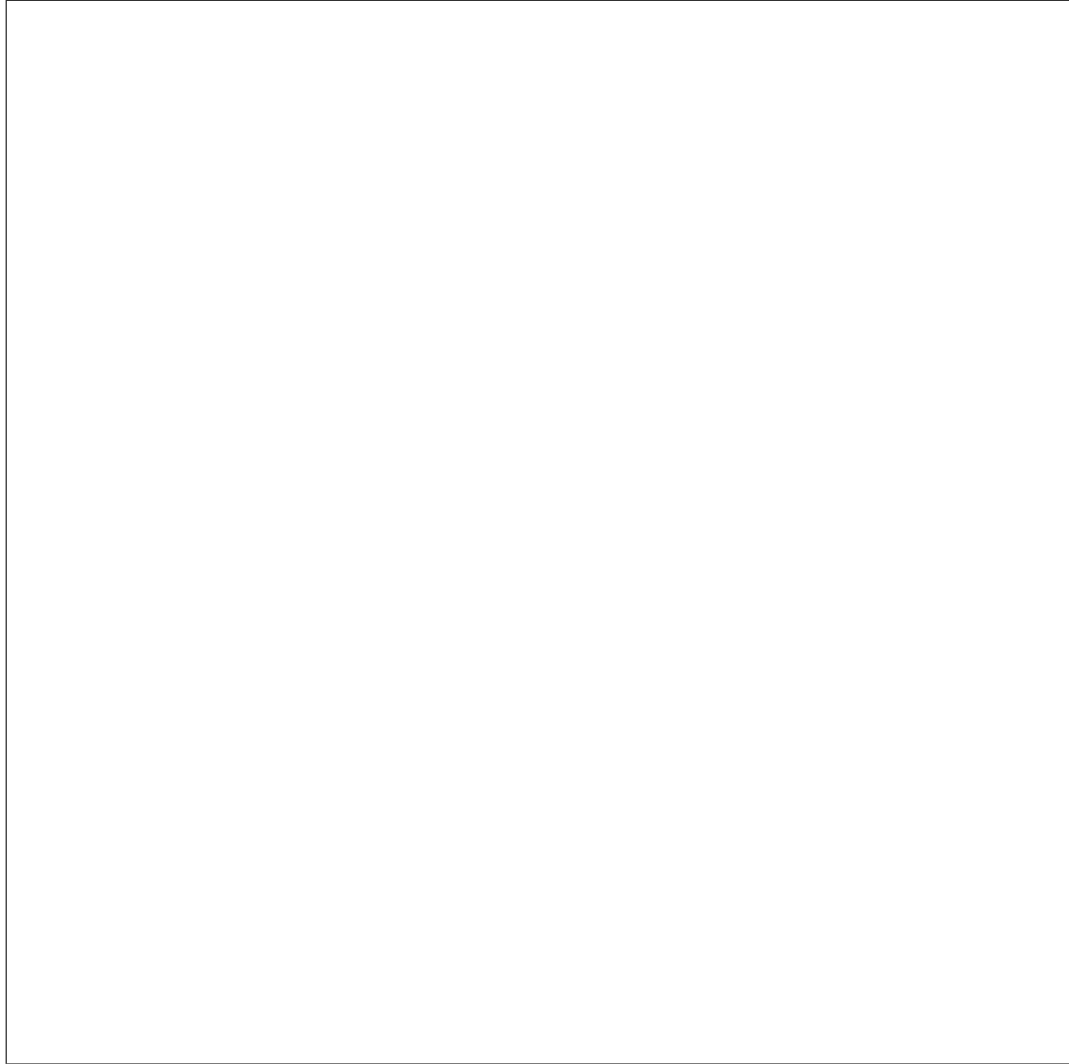
Now conformal mappings give us a way to compare complex domains in a new way.

Question 1. Construct a conformal mapping from the (open) unit disk D_1 to the disk D_2 of radius 2.

Question 2. Let $H = \{z \mid \text{Im}(z) > 0\}$ be the upper half-plane. Show that for any fractional linear function $f = \frac{az+b}{cz+d}$ such that $a, b, c, d \in \mathbb{R}$ are real parameters and the determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is positive (thanks to Ankit for pointing this out), f determines a conformal mapping from H to H .

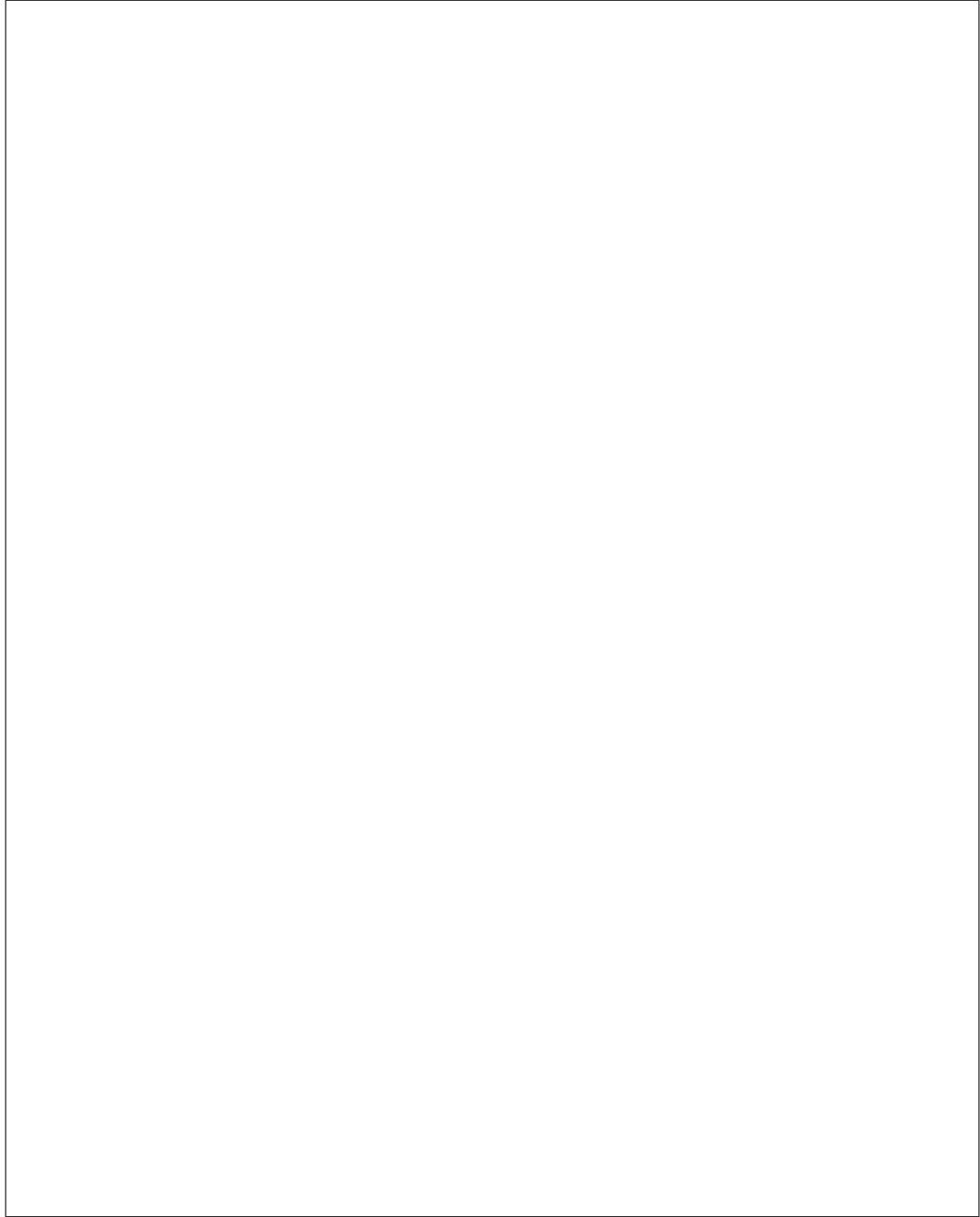


Question 3. *Construct a conformal mapping from the (open) upper half-plane, $H = \{z \mid \text{Im}(z) > 0\}$ to the (open) unit disk, D_1 . Hint: try using a fractional linear transformation (but not with real coefficients in this case).*



Question 4. (a) Construct a conformal mapping $f : S_{(-\pi, \pi)}^{vert} \rightarrow \Omega$ from the vertical complex strip $S_{(-\pi, \pi)}^{vert}$ given by $\{z \mid -\pi < \operatorname{Re}(z) < \pi\}$ and the complement to the ray of negative numbers, $\Omega := \mathbb{C} \setminus \{-r \mid r \in \mathbb{R}_{\geq 0}\}$. (Note: Ω is open, so does not contain 0).

(b) What is the inverse $f^{-1} : \Omega \rightarrow S$? Check that, indeed, this is a holomorphic function with nowhere zero derivative.



Question 5. Construct a conformal mapping $f : S_{(0,\pi)}^{vert} \rightarrow H$ from the vertical complex strip of numbers of real part between 0 and π to the upper half-plane H .

