Worksheet 6: Counting zeroes and poles on functions in a sphere, I.

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1 Conformal functions and the argument principle

Last time we proved the following theorem, which states that the integral of $\frac{f'}{f}$ around a contour is equal to $2\pi i$ times the number of zeroes minus the number of poles of f in this contour, counted with multiplicity (so a zero of multiplicity n contributes $+n$ while a pole of order n contributes $-n$). Today we will combine this with the Riemann sphere S^2.

Recall that if $\Omega \subset \mathbb{C}$ is a domain in $\mathbb{C}$, then a function $f : \Omega \to S^2$ is equivalent to a *meromorphic* function on F on Ω, i.e. a function all of whose singularities are poles, with the added restrictions that all poles are of first order (equivalently, simple) and at points which are not poles, f has non-zero derivative. The functions F and f are related using the stereographic projection: so

$$f(z) = \begin{cases} P_N^{-1}\left(F\left(z\right)\right), & F \text{ defined at } z \\ P_N & z \text{ a pole of } F. \end{cases}$$

Counting zeroes and infinities of a meromorphic function can be translated to counting the number of times the corresponding function *to the Riemann sphere* crosses the North pole N (corresponding to ∞) or the South pole S (corresponding to 0).

Question 1. Suppose F, f are related as above, for $f : \tilde{\Omega} \to S^2$ conformal. Suppose γ is a simple closed curve in $\tilde{\Omega}$ and its interior $\Omega = \text{Int}(\gamma)$ is contained in $\tilde{\Omega}$. Suppose also that F has neither zeroes nor poles on γ (but may have either in its interior).

Show that $\frac{1}{2\pi i} \oint_\gamma \frac{F'}{F}(z)dz$ is equal to $|\Omega \cap f^{-1}(S)| - |\Omega \cap f^{-1}(N)|$, i.e., the number of preimages of the South pole minus the number of preimages of the North pole inside the interior of γ.

2 Holomorphic functions to the sphere

Remember that conformal functions are *almost* the same thing as holomorphic functions: there is just an extra pesky bit involving nonzero derivatives. It turns out that if we allow "singularities" in our conformal condition on functions, i.e. isolated points $p \in \Omega$ where the function is still continuous, but such that f might not preserve angles between curve starting at p, then we get back the condition of holomorphicity. We will not prove this here, but instead use a consequence of such a train of thought as a definition:

Definition 1. *A* holomorphic function *from a domain* $\Omega \subset \mathbb{C}$ *to the sphere* S^2 *is a function* $f : \Omega \to S^2$ *obtained from a meromorphic function* $F : \Omega - \to \mathbb{C}$ *by defining*

$$f(z) := \begin{cases} P_N^{-1} F(z), & F \text{ is defined at } z \\ N, & F(z) = \infty. \end{cases}$$

The difference of this definition from our conformal definition is that we no longer impose any conditions on F other than meromorphicity (so no longer requiring nonzero derivative or simple poles).

Question 2. Show that a holomorphic function f to S^2 defined using definition 1 above is continuous (i.e., $\lim f(z_n) = f(\lim z_n)$, if the limit exists).

Now we can also define a *holomorphic* function from a sphere to a sphere.

Definition 2. *This is a function $f : S^2 \to S^2$ such that the two compositions $f \circ P_N^{-1} : \mathbb{C} \to S^2$ and $f \circ \bar{P}_S^{-1} : \mathbb{C} \to S^2$ are holomorphic (in terms of the above definition).*

Question 3. Show that a holomorphic function f to S^2 defined using definition 2 above is continuous (i.e., $\lim f(\vec{v}_n) = f(\lim \vec{v}_n)$, if the limit exists).

Question 4. Describe the (unique) function $f : S^2 \to S^2$ such that (after converting to a meromorphic function $\mathbb{C} \to \mathbb{C}$ using stereographic projection), the resulting function is $f(z) = z^2$. Show that this function is not conformal precisely at 0 and ∞.