Official Worksheet 5: The Logarithmic derivative and the argument principle.

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1 Definitions and the argument principle

Today's worksheet is a short one: we cover the *logarithmic derivative* and the *argument principle*

Suppose that f is a meromorphic function on a domain Ω (recall: this is a function which is either holomorphic on Ω or is singular with only isolated singularities, all of which are poles.)

Define the *logarithmic derivative*

$$\frac{d\log}{dz} f := \frac{f'}{f}.$$

Note that for any nonzero function f, this definition makes sense everywhere except zeroes or poles of f (where it has at worst poles), since f' is defined everywhere that f is. The notation $d\log$ is indicative of the fact that, where defined, $\frac{d}{dx}\log(f(x)) = \frac{d\log}{dz} f$.

The logarithmic derivative $d\log$ has a number of nice properties. They key property is sometimes known as the *argument principle*, and is as follows.

Theorem 1. *1. If f is nonzero and nonsingular at z_0, then $\frac{f'}{f}$ is nonsingular at z_0.*

2. If f has a pole of order n at z_0 then $\frac{f'}{f}$ has a simple pole with residue equal to $-n$ at z_0.

3. If f has a zero of degree n at z_0 then $\frac{f'}{f}$ has a simple pole with residue equal to n at z_0.

Corollary 1. *If f is a meromorphic function on Ω and γ is a simple closed curve inside Ω such that f is nonsingular and nowhere zero on γ, then $\oint_\gamma \frac{f'}{f}(z)dz$ is equal to $2\pi i(Z-P)$, where Z is the number of zeroes of f inside γ and P is the number of poles inside of γ, both counted* with multiplicity *(i.e. a pole of order n contributes n to P and a zero of degree n contributes n to Z).*

Question 1. (a) Prove Theorem 1. (Hint: look at the Laurent series. You will only need the first nonzero coefficients of f and f'.)

(b) Use the residue formula to prove Corollary 1.

Notice that P and Z are always integers: so the integral in the corollary is always an integer multiple of $2\pi i$. This is very useful in computations: once you compute an contour integral of a logarithmic derivative with error less than π in absolute value, the calculation is complete. This is used, for example, when checking whether there is a zero of the *zeta function* in a given domain: the standard way to do this is to compute a contour integral of the logarithmic derivative to some low precision, as above. In fact this is our best evidence for the Riemann hypothesis, which states that all (positive real value) zeroes of the zeta function lie on the line $\frac{1}{2}+iy$ of complex numbers with real part $1/2$. One can reasonably ask how to distinguish between a zero at $\frac{1}{2}+iy$ and something very close but not quite on the line, like $.500000000001+iy$. As it turns out, the logarithmic derivative is again the answer. The so-called *functional equation* for the zeta function implies that if z is a zero of the zeta function then $1-\bar{z}$ is as well (technically, the functional equation tells you that $1-z$ is a zero, but since the zeta function is real on the real line, we have $\zeta(z)=\overline{\zeta(\bar{z})}$. This means that if $z=.500000000001+iy$ were a zero, then $1-\bar{z}=499999999999+iy$ is a zero as well, so the integral of the function $\frac{\zeta'}{\zeta}$ along a small rectangle about $\frac{1}{2}+iy$ would return $2\cdot 2\pi i$ instead of $1\cdot 2\pi i$. Since all integrals of this form done for y on the order of many milions have returned $1\cdot 2\pi i$, confirming the Riemann hypothesis up to a large distance away from the origin.

2 Examples

Question 2.

1. Compute the logarithmic derivative of e^z.
2. Compute the logarithmic derivative of $\cos(z)$.

3. Compute the logarithmic derivative of $\frac{1}{\cos(z)}$.

4. Compute the logarithmic derivative of $\tan(z)$

5. Compute the logarithmic derivative of the function $\frac{1}{z} + \frac{1}{1+z}$. Where are its poles?

Question 3. Prove that the logarithimc derivative satisfies the following property: $\frac{d\log}{dz}(f \cdot g) = \frac{d\log}{dz} f + \frac{d\log}{dz} g$. In other words, logarithmic derivative converts products to sums. You can either do this directly or start by restricting to the domain where $\log(f)$ and $\log(g)$ are defined, then using uniqueness of analytic continuation.