Official Worksheet 4: Möbius transformations

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1 Introduction

In this section we will analyse conformal functions to the Riemann sphere, and play around a little more with fractional linear transformations (when viewed as functions to and from the sphere, they will be called *Möbius transformations*). We will prove two main theorems. The first will characterize conformal functions from some domain $\Omega \subset \mathbb{C}$ to S^2, and the second will characterize functions from S^2 to S^2.

2 Conformal functions from a complex domain to the sphere

Recall first that a *conformal function* is a function which preserves angles between curves (when we use "function" rather than "transformation", we are not requiring one-to-one or onto properties). A conformal function between two complex domains $f : \Omega \to \Omega'$ is a *holomorphic* function *whose derivative is nowhere zero.* (We proved this a couple of worksheets ago; if you don't remember this, take this as an equivalent definition).

How can we construct conformal functions from some $\Omega \subset \mathbb{C}$ to the sphere S^2? First, remember that the sphere S^2 is almost the same thing (from the conformal point of view) as the complex plane. Indeed, as soon as we remove the north pole, $N = (0, 0, 1) \in S^2$, there is a conformal bijection (one-to-one map which is conformal) between $S^2 \setminus \{N\}$ and the complex plane $\mathbb{C}$, namely the *stereographic projection* (from the North pole), $P_N : S^2 \setminus \{N\} \to \mathbb{C}$. Its (compositional) inverse function $P_N^{-1} : \mathbb{C} \to S^2$ is also conformal. This means that we can get a function $\Omega \to S^2$ that *misses the North pole* by just taking any conformal function $\Omega \to \mathbb{C}$ (equivalently, holomorphic function on Ω with nowhere zero derivative) and applying the composition $\Omega \xrightarrow{F} \mathbb{C} \xrightarrow{P_N^{-1}} S^2$. Conversely, given a function $f : \Omega \to S^2$ that misses the north pole, we can define a conformal function $\Omega \to \mathbb{C}$ as $F := P_N \circ f$. So conformal functions $f : \Omega \to S^2$ that miss the North pole are precisely the same as conformal functions $f : \Omega \to \mathbb{C}$. What if we drop the condition of missing the North pole? Well, from the point of view of stereographic projection, the North pole is the limit of points that project to further and further away in $\mathbb{C}$, i.e., it should be interpreted as ∞. Therefore when trying to interpret the "stereographic projection" of a function $f : \Omega \to S^2$ which sometimes hits the north pole, we need to allow the holomorphic function $F : \Omega \to \mathbb{C}$ to have values equal to ∞ : i.e., poles. A holomorphic function F which is defined on all of Ω except possibly a set of isolated singularities $z_0, z_1, \ldots,$ *which are poles* (i.e., have locally holomorphic inverse) is called *meromorphic*. So we should expect conformal functions $\Omega \to S^2$ to correspond to certain meromorphic functions F on Ω. Where F is finite, we still need to impose the condition $F' \neq 0$ (necessary for conformality). The analogous condition for poles turns out to be simply the requirement that poles of F be *simple*, i.e. first order poles.

This statement is captured in the following theorem.

Theorem 1. *Suppose $\Omega \subset \mathbb{C}$ is an open domain. Suppose F is a meromorphic function (i.e. holomorphic with possible poles) on Ω with poles at $z_0, z_1, z_2, \ldots$ the following properties.*

1. *If $z \in \Omega$ is not a pole, $F'(z) \neq 0$ (non-zero derivative).*
2. *Each pole $z_0, z_1, \ldots$ in Ω is* simple.

Then, the function $f : \Omega \to S^2$ defined as follows:

$$f(z) := \begin{cases} P_N^{-1}\big(F(z)\big), & z \neq z_k (\forall k) \\ N, & z \in \{z_1, z_2, \ldots\}. \end{cases}$$

is conformal. Moreover, any conformal function $f : \Omega \to S^2$ can be obtained in this way from a meromorphic F as above.

We start by proving the converse direction, which we will do in two steps.

Question 1. **(a)** Suppose that $f : \Omega \to S^2$ is a conformal function. Let $z_0, z_1, \cdots \in \Omega$ be all preimages of $N \in S^2$. Let $\Omega' = \Omega \setminus \{z_0, z_1, \ldots\}$ be the complement to these. Show that the function $F := P_N \circ f \mid_{\Omega'}$ (the notation $f \mid_{\Omega'}$ is the *restriction* of f to Ω') is an (everywhere defined) conformal function $\Omega' \to \mathbb{C}$. You may use that the composition of two conformal functions is conformal (indeed, if application of each one preserves angles, application of both together will still preserve angles). In other words, F is holomorphic on Ω' and doesn't have any values with derivative 0 (on Ω').

(b) Let $w_0, w_1, \cdots \in \Omega$ be points such that $f(w_k) = S$ is the South pole $S = (0, 0, -1) \in S^2$. Clearly, none of the z_k's are equal to any of the w_j's (since the value of f cannot be both N and S at the same time), so F is defined on w_k. Show that w_k are precisely the points in Ω' satisfying $F(w_k) = 0$.

(c) Let $\Omega'' = \Omega \setminus \{w_1, w_2, \ldots\}$ be the set of points whose image under f does not hit the South pole. Show that the function $G : \Omega'' \to \mathbb{C}$ defined by conjugate

Southern stereographic projection[1], $G(z) = \bar{P}_S f(z)$ is defind and conformal for all $z \in \Omega''$ and $G(z) = \frac{1}{F(z)}$, (the multiplicative inverse) where both are defined. So G is a holomorphic function on Ω'' with no zero derivatives.

(d) Deduce that the *multiplicative* inverse of F has removable singularities at all singular points of F, i.e. all singularities of F, are *poles* (and not essential singularities). Deduce that all poles of F are simple (first-order) from the fact that G cannot have derivative zero at any of the z_k.

And now we will show the converse:

Question 2. Let $z_0, z_1, \dots \in \Omega$ be a collection of isolated points, and set $\Omega' = \Omega \setminus \{z_1, z_2, \dots\}$. Suppose F is a function on $\Omega' \subset \Omega$ which has nowhere zero derivative in Ω', and such that the singularities at $z_1, z_2, \dots,$ are simple poles. Show that the function

$$f(z) := \begin{cases} P_N^{-1} F(z), & z \neq z_k \forall k \\ N, & z = z_k \end{cases}$$

is conformal. Hint: it's enough to check conformality for each point $z \in \Omega$ independently. Do this first for $z \neq z_k$, and then use the function $G = F^{-1}$ (and the Southern stereographic projection $\bar{P}_S$) to show that the function f defined above remains conformal at the z_k.

3 Conformal functions from the sphere to itself

Now we will look at conformal functions $g : S^2 \to S^2$.

Question 3.

(a) Suppose that g is a (bijective) conformal mapping $S^2 \to S^2$. Show that the function $f_N := g \circ P_N^{-1} : \mathbb{C} \to S^2$ is conformal. (Here the domain is $\Omega = \mathbb{C}$, the whole plane.)

(b) Show that the function $f_S := g \circ \bar{P}_S^{-1} : \mathbb{C} \to S^2$ is also conformal, and satisfies $f_S(z) = f_N(z^{-1})$, when both are defined. Deduce that the (partially defined) functions $F_N := P_N \circ f_N$ and $F_S := P_N \circ f_S$ are meromorphic functions which are *conformal*, i.e. with nonzero derivative at all finite values and with first-order poles at "infinite" values.

(c) Deduce that in the $z \to 0$ limit, we have $\lim_{z \to 0} F_S(z) \cdot z$ is a well-defined complex number, possibly equal to 0 (hint: look at the Laurent series). Deduce that $\lim z \to \infty \frac{P_N \circ f_N(z)}{z}$ also converges to a constant.

[1] There was an error in the initial notes: since they are "looking at the plane from opposite sides", Northern and Southern stereographic projection differ by a map which is not quite conformal, but the mirror image to a conformal map. So the conformal version of Southern stereographic projection is $\bar{P}_S$, given by taking stereographic projection from the South pole, then applying the conjugation map.

(d) If you haven't already, do the Bonus question (Question 3) from Official Worksheet 1. Use the result to deduce that if $g(N) \neq N$ (i.e., if $F_S(0) \neq \infty$) then $F_N := P_N \circ f_N$ above is a fractional linear transformation. Hint: by bijectivity, there is some $p \in S^2 \setminus N$ with $f(p) = N$. See that $P_N(p)$ will be the unique pole of F_N.

Now use (c) to deduce that if $g(N) = N$ then F will be a non-constant linear function, i.e. still a fractional linear transformation with constant denominator.

Now F is uniquely determined by f_N and continuity (continuity implies that the North pole has to go to the limit, $\lim_{z\to\infty} f_N(z)$, where the limit is taken along any sequence $z_1, z_2, \dots,$ approaching ∞ in absolute value). Deduce that conformal functions

$$g : S^2 \to S^2$$

precisely correspond to fractional linear transformations.

When viewed as functions $S^2 \to S^2$, fractional linear transformations are called *Möbius transformations.*

4 Examples

Question 4. (a) Show that the fractional linear transformation $F(z) = 1/z$ corresponds to the 180-degree rotation of the the sphere around the x axis, $(x, y, z) \mapsto (x, -y, -z)$. (In particular it sends the North pole, ∞, to the South pole, 0, and vice versa.)

(b) Show that the fractional linear transformation $F(z) = iz$ corresponds to rotation of the sphere by 90 degrees around the y axis, $(x, y, z) \mapsto (-y, x, z)$.

(c) The above two transformations are *rigid transformations of the sphere* (they preserve distances between points, equivalently are given by an orthogonal matrix). It can be checked that any orientation-preserving (equivalently, determinant-one) orthogonal matrix does indeed induce a conformal map from S^2 to S^2. However the converse is not true, as we shall see.

Check that the conformal map $S^2 \to S^2$ corresponding to the function $F(z) = z + 1$ does not preserve distances between points on S^2.

Question 5. (This one is a little more time-consuming, but fun to do when you have some free time.)

(a) Convince yourself that stereographic projection takes circles on the sphere (equivalently, intersections of the sphere with a plane) to *circles and lines* on the plane $\mathbb{C}$.

(b) Prove that any fractional linear transformation takes circles and lines on $\mathbb{C}$ to circles and lines (hint: it's enough to do this for translations, $z \mapsto z + \lambda$ and the inversion map $z \mapsto z^{-1}$, then compose these to get any fractional linear transformation). Give an example of a fractional linear transformation that takes a circle to a line, and one that takes a line to a circle. In fact this is a way of characterizing conformal transformations from the Riemann sphere to itself: it is precisely those (continuous) transformations that take circles to circles.