Official Worksheet 3: The Riemann Sphere

April 3, 2020

1 Introduction

We have worked so far with holomorphic functions from the complex plane $\mathbb{C}$ (or regions thereof) to itself. It turns out that holomorphic functions also work as functions from (or to) the sphere S^2 (called the Riemann sphere in this context). Here the sphere is defined as the set $S^2 := \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ for x, y, z *real* coordinates (this is a subset of the real Euclidean space, $\mathbb{R}^3$). It is important to remember that, while the sphere is a three-dimensional object (it is a manifold in 3-space), it "locally" looks two-dimensional (this is why maps of the Earth can be drawn on a piece of paper). So a function on a sphere can be defined, for example, as a pair of functions on each hemisphere, and each hemisphere can be flattened out to a disk in the plane $\mathbb{R}^2$ (or $\mathbb{C}$). We will use a variant of this point of view to define holomorphic functions on the sphere. However, we will start with defining *conformal* functions on the sphere.

2 Smooth paths on a sphere and angles

In this class, a smooth *path* (or *curve*) on S^2 is a function $\gamma : [0, T] \to \mathbb{R}^3$ such that $\gamma(t) \in S^2$ for any $t \in [0, T]$, and such that γ is differentiable as a function on $[0, T]$, with continuous and *nonzero* derivative. (Note: in 141, usually smoothness requires infinitely many derivatives and we do not require the derivative to be nonzero... here we're following notation from Stein.)

A useful property of functions to S^2 (that many of you know from differential topology or multivariable calculus) is that, if $\gamma : [0, T] \to S^2$ is a smooth path and $\gamma(t) = \vec{p}$, some point of S^2, then the derivative $\dot{\gamma}(t)$ is in the *tangent plane* to S^2 at $\vec{p}$, which is the space of orthogonal vectors to $\vec{p}$, i.e. $\{\vec{v} \in \mathbb{R}^3 \mid \vec{v} \perp \vec{p}$. If you haven't seen this, convince yourself that this is the case:

Question 1. Show that if γ is a smooth curve on S^2 then its derivative vector $\dot{\gamma}(t) \in \mathbb{R}^3$ is orthogonal to the vector $\gamma(t)$. (Hint: one way to do this is to use the fact that for $F(x, y, z) = x^2 + y^2 + z^2$, the composition $f \circ \gamma$ has zero derivative – why?)

Let $\Omega \subset \mathbb{C}$ be an open domain and let $\Omega' \subset S^2$ be an open domain in a sphere, as above (an open domain in a sphere is the complement to a closed subset of the sphere, or equivalently, the intersection of a sphere with an open domain in $\mathbb{R}^3$). Suppose $f : \Omega \to \Omega'$ and $g : \Omega' \to \Omega$ are functions.

Then f is called *conformal* if for any smooth path γ in Ω (with nowhere zero derivative) the derivative of the image path $\tilde{\gamma} = \gamma \circ f$ is again nonzero everywhere, and f preserves angles, i.e., if γ_1, γ_2 are two curves in S^2 with the same starting point $p \in \Omega$, then the angle between the tangents to γ_1 and γ_2 is equal to the angle between $f \circ \gamma_1$ and $f \circ \gamma_2$. Similarly, g is called conformal if for two curves $\tilde{\gamma}_1, \tilde{\gamma}_2$ *on the sphere* both starting at the same point $\vec{p} \in S^2$, the resulting image curves $g \circ \tilde{\gamma}_1$ and $g \circ \tilde{\gamma}_2$ are at the same angle from each other as $\tilde{\gamma}_1, \tilde{\gamma}_2$. There is a little bit of a sign issue with defining angles on curves: you get opposite angles if you are looking at the sphere "from the outside" vs. "from the inside". We will use the outside perspective.

Here is the official definition of angle we will use: given two vectors $\vec{v}, \vec{w}$ in $\mathbb{R}^3$, the cosine of the angle $\cos(\Theta_{\vec{v},\vec{w}})$ is defined as the dot product of the two corresponding unit vectors, with the following formula:

$$\cos(\Theta_{\vec{v},\vec{w}}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}| \cdot |\vec{w}|}.$$

This determines the angle up to sign; the sign of the angle can be recovered by remembering that we are measuring angles at a point of the sphere $\vec{p} = \gamma_1(0) = \gamma_2(0)$. The angle is then positive (i.e., $0 < \Theta < \pi$) if the cross product $\vec{v} \times \vec{w}$ is positively proportional to the point of the circle $\vec{p}$, and negative (i.e., $\pi < \theta < 2\pi$) if the cross product is negatively proportional to the point of the circle $\vec{p}$. We will mostly not worry about signs, and just check that the cosine of an angle is preserved.

3 Stereographic projection.

As it turns out, there are no conformal maps from S^2 to a domain in $\mathbb{C}$ (essentially for the reason that you cannot squash a globe onto a single piece of paper without creating creases). However, there are many interesting maps from (domains in) $\mathbb{C}$ to S^2. In fact, the standard map projection on most classroom walls sends a rectangle with dimensions $2\pi \times \pi$ onto the sphere, in a stereographic way! (Although this map does not preserve distances, the fact that it preserves angles was invaluable to early modern sailors: it makes it easier to know how much you need to turn your ship to follow a course on a map.)

There are in particular two special maps of this type taht we will use, namely, the *stereographic maps.* These are the inverse stereographic projection from the *North pole* (written P_N^{-1} and the inverse stereographic projection from the South pole (written P_S^{-1}). Here the -1 powers indicate that these are the *inverse* (by composition) maps to stereographic projection.

Definition 1. *We define the north pole to be the point* $(0,0,1) \in S^2$ *and the South pole to be the point* $(0,0,-1) \in S^2$.

Definition 2. *Let* $\vec{p} = (x,y,z)$ *be a point of the sphere distinct from* $N = (0,0,1)$. *The northern stereographic projection* $P_N(\vec{p})$ *is defined to be the point where the line* $\overline{N, \vec{p}}$ *intersects the* x, y*-plane. As a formula:*

$$P_N(\vec{p}) = \frac{x}{1-z} + i\frac{y}{1-z}.$$

Similarly, southern stereographic projection is the same thing, with "north" replaced by "south". For $\vec{p} = (x,y,z)$ *a point of the sphere distinct from* $S = (0,0,-1)$, *and given by the formula*

$$P_S(\vec{p}) = \frac{x}{1+z} + \frac{y}{1+z}i.$$

Stereographic projections give bijections between the sphere (without the north or South pole) and the plane.

Question 2. Define the inverse stereographic projection to be the function

$$P_N^{-1} : X + Yi \mapsto \left(\frac{2X}{1+X^2+Y^2}, \frac{2Y}{1+X^2+Y^2}, \frac{-1+X^2+Y^2}{1+X^2+Y^2} \right).$$

Convince yourself that $P_N : S^2 \setminus \{N\} \to \mathbb{C}$ and $P_N^{-1} : \mathbb{C} \to S^2 \setminus \{N\}$ are inverse maps of sets.

Question 3. Here we will show that (Northern) stereographic projection preserves angles up to sign, i.e. that if γ_1, γ_2 are two paths on S^2 both starting at a point $p = (x, y, z)$ then their angle is preserved. If you get stuck here or this is taking too much time, move to the next question.

(a) Remember that if γ_1, γ_2 are both smooth paths on the sphere starting at some poing $\vec{p} = (x, y, z)$ on the sphere, then their derivatives are in the plane orthogonal to $\vec{p}$. The stereographic projection multiplies the two tangent vector $\dot{\gamma}_1(0), \dot{\gamma}_2(0)$ by the *complex* vector

$$\left(\frac{\partial P_N}{\partial x}(\vec{p}), \quad \frac{\partial P_N}{\partial y}, \quad \frac{\partial P_N}{\partial z} \right).$$

(This is the “complex Jacobian” of this transformation). We need to show that this matrix takes pairs of vectors *in the plane orthogonal to* $\vec{p}$ to angles in $\mathbb{C}$ forming the same angle.

First, show this is the case in the special case $\vec{v} = S = (0, 0, -1)$ is the *south pole* (hint: in this case the Jacobian has a particularly nice form).

(b) Next, use polar coordinates to show that (northern) stereographic projection preserves angles at points of S^2 which are not N, S. (This is a bit of a calculation, you might want to skip it the first time around.)

The same thing can be proven for Southern stereographic projection, in the same way. Since P_N, P_S are bijective maps that preserve angles, the same must be true of their inverse[1].

4 Another characterization

Now we have a new way of thinking about conformal maps from a region of the circle, $\Omega' \subset S^2$. Indeed, suppose $\tilde{\Omega} \subset S^2$ is a domain that does not contain the North pole, and $\Omega \subset \mathbb{C}$ is a domain in the complex plane. Then $f : \tilde{\Omega} \to \Omega$ is conformal if and only if the composition $f \circ P_N^{-1}$ is a conformal map from $P_N^{-1}(\tilde{\Omega}) \to \Omega$ between domains of the complex plane! Similarly if $\tilde{\Omega}$ doesn't contain the South pole. This lets us check conformal properties of maps to and from the sphere in terms of conformal properties of maps between complex domains: i.e., holomorphic maps with nonzero derivative.

5 For next time

We will characterize all conformal maps from S^2 to S^2, and we will see (using stereographic "maps" of the sphere by $\mathbb{C}$) that they precisely correspond to fractional linear transformations! In particular, there are not that many of them. If you want to get a head start, you already have all the tools to work this out.

[1]There are some technical differentiability checks I'm ignoring here.