Official Worksheet 2: Conformal mappings.
(Material from this worksheet may be on exam!)

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1 Introduction

In this worksheet we will define and study conformal maps between two-dimensional domains, which are *maps which preserve angles.* First, we will do a warm-up, and study rigid transformations of the plane, which are *maps which preserve distances.* If you have seen rigid transformations, you can skip to section 3.

1.1 Functions as mappings

Here (and for the next several lectures), the way we will be thinking of a function $f : A \to B$ between two sets (usually, domains in $\mathbb{C}$) is as a *mapping* or (another word for the same thing) as a *transformation.* This is a rule that "moves" every point of A to a point of B, thought of as a geometric motion. Think of pinching and rotating a Google Map or a picture on your phone (in fact, conformal maps are precisely maps from the plane to itself that locally look like pinch-and-zoom transformations, up to terms of order ≥ 2).

2 Rigid transformations

A function $f : \mathbb{C} \to \mathbb{C}$ is a *rigid transformation* if it is continuous and bijective (i.e., one-to-one and onto) and if for any two points z, z' the distance between the transformed points $|f(z) - f(z')| = |z - z'|$, equals the distance between the original two points.

There are three key examples of rigid transformations.

- *A (vector) translation*
$$f(z) = z + z_0,$$
for $z_0 = a + bi$ a fixed complex number.
- *A rotation*
$$f(z) = e^{i\theta} z,$$
for θ a fixed angle.

- *A reflection,* the key example being

$$f(z) = \bar{z} (= x - yi),$$

reflecting a complex number over the real axis.

(Quick exercise: if it is not obvious to you or you want to check, convince yourself that these preserve differences, by computing $|f(z) - f(z')|$.)

Notice that the third example is different from the other two in that it is not a holomorphic function.[1] It also reverses orientation (i.e., the notions of left/right and clockwise/counterclockwise get switched by a reflection, but not by a translation or a rotation).

Question 1.
(a) Show that a complex number is determined by its distances to 0 and to 1 *up to conjugation.* Concretely, show that if z is a complex number and z' is another number such that $|z' - 0| = |z - 0|$ and $|z' - 1| = |z - 1|$ then either $z' = z$ or $z' = \bar{z}$. (BTW, geometrically this the "sss" point of view on triangles: saying that a triangle is determined up to orientation by the lengths of the three sides.)

(b) Suppose $f : \mathbb{C} \to \mathbb{C}$ is a rigid transformation (remember: not necessarily holomorphic!) and say that $f(0) = z_0, f(1) = z_1$. Show that $f(1) = z_0 + e^{i\theta}$.

[1]It is in fact antiholomorphic.

(c) Let z be another complex number, with $a = |z - 0|, b = |z - 1|$. Show that if a point $\tilde{z}$ has distance a from z_0 and b from z_1 then either $\tilde{z} = z_0 + e^{i\theta}z$ or $\tilde{z} = z_0 + e^{i\theta}\bar{z}$.

(d) Deduce that every rigid transformation is either of the form $f : z \mapsto z_0 + \exp(i\theta)z$ or of the form $f : z \mapsto z_0 + \exp(i\theta)\bar{z}$, for θ and angle and z_0 a constant. This implies that every rigid transformation is the composition of a rotation and a rotation, or a composition of a translation, a rotation, and a conjugation. (Hint: you know from (c) that this formula holds for any specific z, and moreover you know that θ, z_0 are independent of z, since they are determined by $f(0), f(1)$. It remains to check that the choice of "to conjugate or not to conjugate" is the same for each z. Do this for example by using that once you know $f(0), f(1)$ and also $f(i)$, every point is uniquely determined by its distance to the three vertices of a triangle.)

3 Conformal maps

Say that Ω, Ω' are regions in $\mathbb{C}$. We say that a map $f : \Omega \to \Omega'$ is *conformal* if it is a bijection, it is differentiable with continuous derivative, and it *preserves angles*.

Now we know how to take angles between lines, but since f might take a line to a more complicated (but still smooth!) curve, we need to understand how to measure angles between curves (i.e., paths).

Definition 1. *Assume γ_1, γ_2 are smooth paths starting at z_0. Then the* angle $\theta_{\gamma_1,\gamma_2}$ *between γ_1 and γ_2 at z_0 is the angle between the (nonzero, since γ is assumed smooth!) vectors $\dot{\gamma}_1(0)\big(= \frac{d\gamma_1}{dt}(0)\big)$ and $\dot{\gamma}_2(0)\big(= \frac{d\gamma_2}{dt}(0)\big)$.*

Remember that for us, angles are oriented (the angle between the vectors 1 and i is $\pi/2$, whereas the angle between the vectors i and 1 is $3\pi/2$). One way to measure an angle between complex numbers is to take the argument,

$$\theta_{\gamma_1,\gamma_2} = \mathrm{Arg}\left(\frac{\dot{\gamma}_2(0)}{\dot{\gamma}_1(0).}\right)$$

Now we can define conformal maps:

Definition 2. *A map $f : \Omega \to \Omega'$ is conformal if for any $z_0 \in \Omega$ and any two smooth paths γ_1, γ_2 (functions from $[0,T] \to \Omega$ with everywhere nonzero derivative) starting from z_0, it is the case that $f \circ \gamma_1, f \circ \gamma_2$ have everywhere nonzero time derivatives and*

$$\Theta_{\gamma_1,\gamma_2} = \Theta_{f\circ\gamma_1, f\circ\gamma_2}.$$

In other words, f preserves angles between curves.

Notice that (in the plane), a curve $\gamma(t)$ from z_0 can be approximated to first order by the line $L(t) = z_0 + t\dot{\gamma}(0)$, with error $o(t)$ (small compared to t). This means that it is enough to check the conformal property for curves γ which are line segments (but $f \circ \gamma$ may no longer be a line segment).

Question 2. Show that translations $z \mapsto z + z_0$ and scalings, $z \mapsto \lambda z$ (for $z_0 \in \mathbb{C}$ any complex number and $\lambda \in \mathbb{C} \setminus \{0\}$ a nonzero complex number) are conformal maps from $\mathbb{C}$ to $\mathbb{C}$.

4 Conformal functions are holomorphic!

It turns out that conformal functions are holomorphic. And conversely, in order to be conformal a holomorphic function between domains $\Omega \to \Omega'$ needs to be bijective (one-to-one and onto) and have nonzero derivatives (intuitively: having a zero derivative might "crush" a curve and make its derivative zero, but having a nonzero derivative pinches and zooms it, and this does not change angles.)

This sounds like it might be hard to prove, but in fact it is an immediate consequence of Cauchy-Riemann. The thing to remember is that the derivative of a holomorphic function f along a path γ starting at z_0 is equal to $f'(z_0)$

times the "complex speed" of γ, i.e., if f is holomorphic and γ is smooth then $\frac{d(f\circ\gamma)}{dt} = f'(z_0)\cdot\dot{\gamma}(0)$. It now follows that (if $f'(z_0)\neq 0$), the effect of f on the tangents to curves at 0 is multiplying by $\lambda := f'(z_0)$. This preserves angles, as you've checked before. Conversely, you can observe that f being conformal implies that $\frac{df}{dx} = i\cdot\frac{df}{dy}$, and use Cauchy-Riemann. Thus conformal maps are holomorphic. The other conditions of conformality (being bijective and taking curves with nonzero derivative to curves with nonzero derivative) then imply that a holomorphic function $f:\Omega\to\Omega'$ is a conformal mapping if and only if f is bijective and has everywhere nonzero derivative. In fact, the second condition is not necessary:

Question 2. **(a)** Show that if Ω,Ω' are open domains in $\mathbb{C}$ and a function $f:\Omega\to\Omega'$ is holomorphic and a bijection, then it is not the case that $f'(z_0)=0$ for any $z_0\in\Omega$. (Hint: Assume $f'=0$. Hint: assume that $f(z)-f(z_0)$ is a zero of degree n. Since $f'=0$, we must have $n\geq 2$. Now write $f(z)-f(z_0) = a_n\Delta z^n + a_{n+1}\Delta z^{n+1}+\ldots$. Show that there exists a power series $g(z)$ with $g(z) = z+O(z^2)$ and $f(z)-f(z_0) = a_n g(z)^n$. It follows from the identities $g(z_0)=0$ and $g'(z_0)\neq 0$ (for example from the intermediate value theorem) that for any (think small) disk $D_r(z_0)$ around z_0 in Ω, there exists a disk $D_\epsilon(0)$ around 0 such that every complex number $\alpha\in D_\epsilon$ has a preimage, $z\in D_r$ with $g(z)=\alpha$. Now multiply α by the root of unity ζ_n to see f is not one-to-one.

5 Examples of conformal maps

Question 3. Show that the following functions between the following domains are conformal mappings from domain Ω to domain Ω'.
(a) $f(z) = 1/z$, with $\Omega = \Omega' = \mathbb{C} \setminus \{0\}$.

(b) $f(z) = \exp(iz)$, with $\Omega = \{z \in \mathbb{C} \mid -\pi < \mathrm{Re}(z) < \pi\}$ the vertical strip between $-\pi$ and π and $\Omega' = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ the complement to the ray consisting real numbers ≤ 0.

(a) $f(z) = 1/(1-z)$, with $\Omega = D_1$ the open unit disk and $\Omega' = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 1/2$ the half-plane bounded on the left by the vertical line $\mathrm{Re}(z) = 1/2$

6 For next time

If you want to get a head start on next time's worksheet, look up the stereographic projection, a bijection between the sphere with the North (or South) pole removed and the Euclidean plane $\mathbb{R}^2$. Try to show this mapping preserves angles (i.e., is conformal, in a higher-dimensional sense). This is more concrete for the inverse: so an exercise we will set up next time is to check that the map from the plane $\mathbb{R}^2$ to the punctured sphere $S^2 \setminus \{0, 0, 1\} \subset \mathbb{R}^3$ which is inverse to stereographic projection takes pairs of intersecting lines (or more generally, curves) to pairs of curves (now in 3-space!) at the same angle. This will allow us to study *conformal*, equivalently, holomorphic functions whose domain or range is the Riemann sphere (also known as the Riemann Sphere). Fractional linear transformations are the key examples of conformal mappings from the sphere to the sphere.