Worksheet 2: beginning the Weierstrass product formula

This worksheet is optional, material from it will not be on the final.

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1 Introduction

If two polynomials have the same roots (with the same multiplicities) then they differ by a constant factor. This means that a polynomial with $f(z)$ with all roots simple (zeroes of order one) can be expressed as a constant times the product $(z-\lambda_1)\cdot\dots\cdot(z-\lambda_n)$. (If you have not seen this fact, we will re-derive it later).

In this worksheet we will show that something like this is also true for holomorphic functions with infinitely many roots, so long as they have bounded growth as z approaches ∞.

2 Nowhere zero entire functions are exponents.

Let f be an entire function with *no zeroes* in the complex plane. A standard example of such a function is $\exp(z)$. Another example is $\exp(z^2)$. We will show in this section that all examples are of this type.

Theorem 1. *Assume f is a function which is entire (defined and holomorphic on all of $\mathbb{C}$) and assume f has no zeroes (equivalently, $1/f$ is also entire). Then there exists an entire function $g(z)$ such that $f(z) = \exp(g(z))$.*

Proof. One approach is to try to take $g(z) = \log f(z)$. However, this has a problem: the log function is not defined (or at least not holomorphic) at negative real numbers, and $f(z)$ may take negative real values (for example, for $f(z) = \exp(z)$, we have $\exp(\pi i) = -1$). However there is a better approach: remember that the essential difficulties with defining the logarithm have to do with the fact that its derivative, $1/z$, has a residue (and so $\log(z)$ accumulates a "shift" by $2\pi i$ whenever you try to go around a loop around the origin). The function g will also be defined an antiderivative, but now of a function which is entire.
Question 1. Let Ω be the domain on which $f(z)$ does not take negative real values.

(a) Show that Ω contains a disk (for instance it's enough to show it is open and nonempty).

(b) Show that on Ω, the function $g(z) = \log(f(z))$ is defined, holomorphic, and satisfies the differential equation $g'(z) = \frac{f'(z)}{f(z)}$. Because of what you just showed, the function

$$\frac{f'(z)}{f(z)}$$

is called the *logarithmic derivative* of $f(z)$. It is defined everywhere where f is defined and nonzero: since for us f is entire with no zeroes, its logarithmic derivative is also entire.

(c) Show that $g(z) = \log f(z)$ defined on Ω as above has a holomorphic analytic continuation to all of $\mathbb{C}$ (use the fact that entire functions have an antiderivative, and be careful about the free constant in the antiderivative expression).

3 Product formula for the sin function

Now we apply the above principle to give a product expansion for the sine function. This expansion can be used to compute all zeta function values for which there exists a closed formula!

Namely, let $f(z)$ be the rescaled sine function, $f(z) = \sin(2\pi z)$. We know from class and homework that the roots of this function are all integers $n \in \mathbb{Z}$. We would like to express $f(z)$ as a product of polynomial functions, with the same roots. One could try to guess a formula for $f(z)$ in terms of the product $\prod_{n\in\mathbb{Z}} z - n$, but this product will not converge anywhere (indeed, for any fixed z the terms go to ∞)! Instead, we opt for a rescaled version: define $\phi(z)$ "=" $\prod_{n\in\mathbb{Z}} 1-\frac{z}{n}$. This doesn't make sense for $n=0$, so we adjust: $\phi(z) = z\cdot\prod_{n>0}(1-z/n)(1-z/(-n))$, where the first factor is responsible for the vanishing at 0. The grouping by putting $+n$ and $-n$ factors together is convenient because the resulting product becomes $\phi(z) = z\cdot\prod_{n\geq 1} 1-\frac{z^2}{n^2}$, which converges (recall that if a_n converges absolutely then $\prod(1-a_n)$ also converges). And by a straightforward analysis argument, ϕ is a holomorphic function with simple zeroes precisely at the integers $n\in\mathbb{Z}$.

Question 2. We will finish this argument in a series of exercises.

Theorem 2. *For* $\phi(z) = z\prod_{n=1}^{\infty}(1-\frac{z^2}{n})$, *we have* $\sin(\pi z) = \pi\phi(z)$.

Note that parts (c)-(e) can be significantly simplified by taking logarithmic derivatives of both sides (see 3.2 in the book), but here I will give a proof that is more useful from the point of view of generalizations.

(a) Using the holomorphicity and vanishing results about ϕ given above, show that $\frac{f(z)}{\phi(z)}$ is an entire function with no zeroes. (For $f(z) = \sin(\pi z)$.)

(b) Deduce that $f(z) = \phi(z)\cdot\exp g(z)$, for $g(z)$ an entire function.

(c) (This is a little more tricky, I'll provide a solution if you are having trouble.) Show that both $f(z)$ and $\phi(z)$ have at worst exponential growth (i.e., there exists a real constant $\beta > 0$ such that $\frac{f(z)}{\exp(\beta|z|)} \to 0$ as $|z| \to \infty$).

(d) Deduce that if $\frac{f(z)}{\phi(z)} = \exp g(z)$ then $g(z) = az + b$ is a linear function.

(e) By looking at the $z \in \mathbb{R}_{\geq 0}, z \to \infty$ asymptotic, deduce that $a = 0$, i.e. $\frac{f(z)}{g(z)}$ is a constant.

(f) By comparing derivatives of the two functions at 0, show that $\phi(z) = \frac{f(z)}{\pi}$. This is the product formula for the sine function.

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