Official Worksheet 1: Fractional Linear transformations. (Material from this worksheet may be on exam!)

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1 Introduction

In this short worksheet we will play around with *fractional linear transformations.* These are functions of the form

$$f(z) = \frac{az+b}{cz+d},$$

where a, b, c, d are complex constants, which have certain nice properties.

2 Fractional linear functions

A *rational function* is a function of the form $f(z) = \frac{p(z)}{q(z)}$, where p and q are both polynomials. A *fractional lienar* function is a rational function $f(z) = \frac{p(z)}{q(z)}$ with the following properties:

1. $p = az + b, q = cz + d$ are either constant or linear polynomials in z.
2. $q(z)$ is not zero (otherwise f would be nowhere defined)
3. $f(z)$ is not constant.

Question 1.
(a) Show that the conditions (2), (3) above on f above are equivalent to the condition that the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has nonzero determinant.

(b) Show that the two functions $f(z) = \frac{az+b}{cz+d}$ and $f'(z) = \frac{a'z+b'}{c'z+d'}$ (satisfying (1)-(3) above) are equal if and only if the matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $M' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ are nonzero scalar multiples of each other (i.e., $M' = \lambda M$ for a nonzero complex number λ).

3 As mappings from the plane with ∞ to itself

We see immediately that a fractional linear function is either entire (when $c = 0$ and the denominator is zero) or (when $c \neq 0$), has one pole at $z_0 = -d/c$, the zero of the denominator). Convince yourself that this is a *simple pole*.

Question 2.

(a) Show that, if $c \neq 0$ and $\frac{az+b}{cz+d}$ is a fractional linear transform then $\lim_{z\to\infty} f(z)$ exists and is equal to a/c.

(b) Extend the function f (a function from a domain of definition to $\mathbb{C}$) to a function from the set $\mathbb{C} \sqcup \{\infty\}$, consisting of all complex numbers and the symbol ∞, by setting $f(\infty) = a/c$ (defined to be ∞ if $c = 0$) and $f(z_0) = \infty$ for z_0 the pole (if it exists). Prove that any fractional linear transformation extended in this way becomes a *one-to-one and onto* (i.e., bijective) function from $\mathbb{C} \sqcup \{\infty\}$ to itself.[1]

4 Compositions

The composition of two fractional linear transformations (as functions from $\mathbb{C} \sqcup \{\infty\}$ to itself) is again a fractional linear transformation, as you can see from the next question.

Question 2. (a) Show that the composition (as functions from the set $\mathbb{C} \sqcup \{\infty\}$ to itself) of $f = \frac{az+b}{cz+d}$ and $f' = \frac{a'z+b'}{c'z+d'}$ is the fractional transformation $f'' = \frac{a''z+b''}{c''z+d''}$, where the matrix

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

[1] Next lecture, we will see that $\mathbb{C} \sqcup \{\infty\}$ can be seen geometrically as a *Riemann sphere*.

(b) Deduce that the inverse function (inverse *in the sense of composition*) to a fractional linear transformation is a fractional linear transformation, and in particular a function which is holomorphic with at most one pole (hint: use the matrix inverse).

5 Bonus questions

Question 3.
(a) What is the residue of the pole of a fractional linear transformation in terms of a, b, c, d?

(b) Show that a function $f(z)$ which is defined on all of $\mathbb{C}$ except a single simple pole, and is bounded when $z \to \infty$, is a fractional linear transformation **Hint:** Subtract the singular part, then use Liouville's theorem.