

# Official Worksheet 1: Fractional Linear transformations.

(Material from this worksheet may be on exam!)

March 30, 2020

## 1 Introduction

In this short worksheet we will play around with *fractional linear transformations*. These are functions of the form

$$f(z) = \frac{az + b}{cz + d},$$

where  $a, b, c, d$  are complex constants, which have certain nice properties.

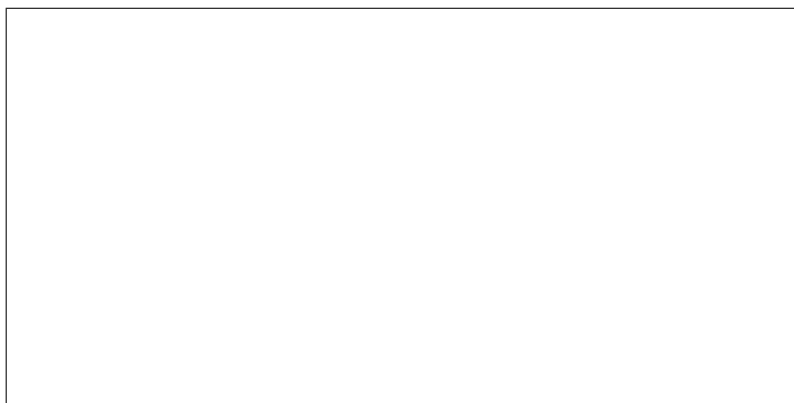
## 2 Fractional linear functions

A *rational function* is a function of the form  $f(z) = \frac{p(z)}{q(z)}$ , where  $p$  and  $q$  are both polynomials. A *fractional linear function* is a rational function  $f(z) = \frac{p(z)}{q(z)}$  with the following properties:

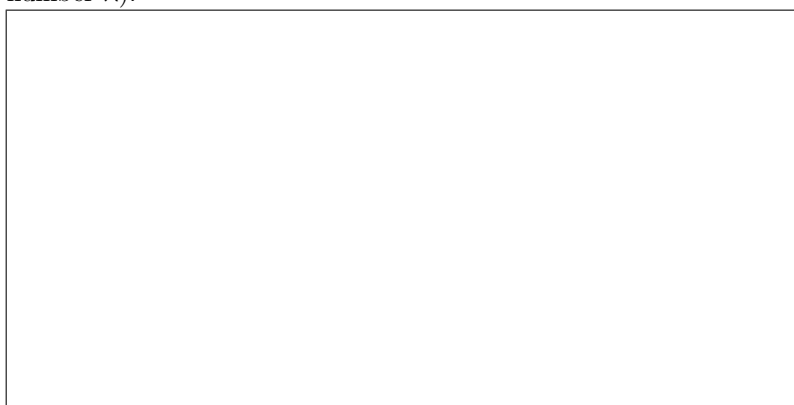
1.  $p = az + b, q = cz + d$  are either constant or linear polynomials in  $z$ .
2.  $q(z)$  is not zero (otherwise  $f$  would be nowhere defined)
3.  $f(z)$  is not constant.

### Question 1.

(a) Show that the conditions (2), (3) above on  $f$  above are equivalent to the condition that the matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  has nonzero determinant.



(b) Show that the two functions  $f(z) = \frac{az+b}{cz+d}$  and  $f'(z) = \frac{a'z+b'}{c'z+d'}$  (satisfying (1)-(3) above) are equal if and only if the matrices  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and  $M' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$  are nonzero scalar multiples of each other (i.e.,  $M' = \lambda M$  for a nonzero complex number  $\lambda$ ).



### 3 As mappings from the plane with $\infty$ to itself

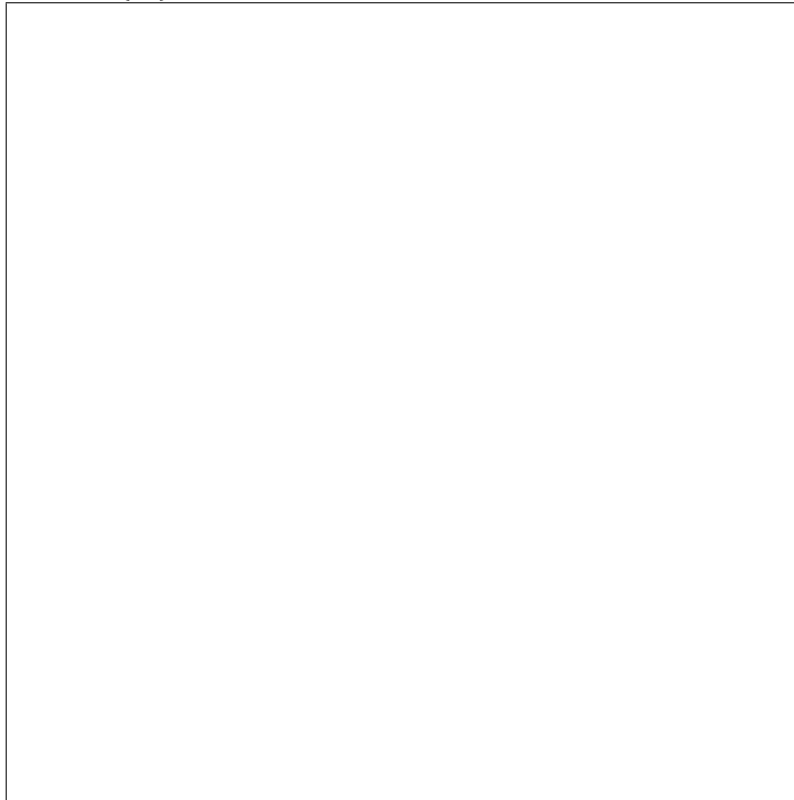
We see immediately that a fractional linear function is either entire (when  $c = 0$  and the denominator is zero) or (when  $c \neq 0$ ), has one pole at  $z_0 = -d/c$ , the zero of the denominator). Convince yourself that this is a *simple pole*.

**Question 2.**

(a) Show that, if  $c \neq 0$  and  $\frac{az+b}{cz+d}$  is a fractional linear transform then  $\lim_{z \rightarrow \infty} f(z)$  exists and is equal to  $a/c$ .

(b) Extend the function  $f$  (a function from a domain of definition to  $\mathbb{C}$ ) to a function from the set  $\mathbb{C} \sqcup \{\infty\}$ , consisting of all complex numbers and the

symbol  $\infty$ , by setting  $f(\infty) = a/c$  (defined to be  $\infty$  if  $c = 0$ ) and  $f(z_0) = \infty$  for  $z_0$  the pole (if it exists). Prove that any fractional linear transformation extended in this way becomes a *one-to-one and onto* (i.e., bijective) function from  $\mathbb{C} \sqcup \{\infty\}$  to itself.<sup>1</sup>



## 4 Compositions

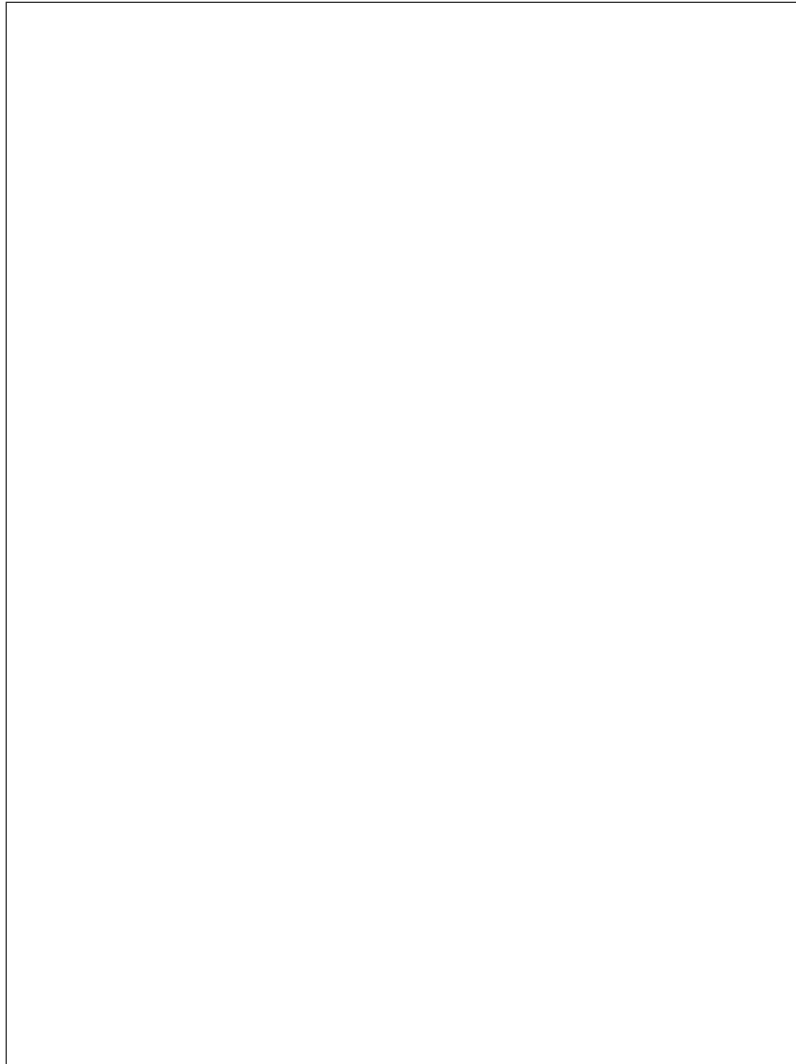
The composition of two fractional linear transformations (as functions from  $\mathbb{C} \sqcup \{\infty\}$  to itself) is again a fractional linear transformation, as you can see from the next question.

**Question 2.** (a) Show that the composition (as functions from the set  $\mathbb{C} \sqcup \{\infty\}$  to itself) of  $f = \frac{az+b}{cz+d}$  and  $f' = \frac{a'z+b'}{c'z+d'}$  is the fractional transformation  $f'' = \frac{a''z+b''}{c''z+d''}$ , where the matrix

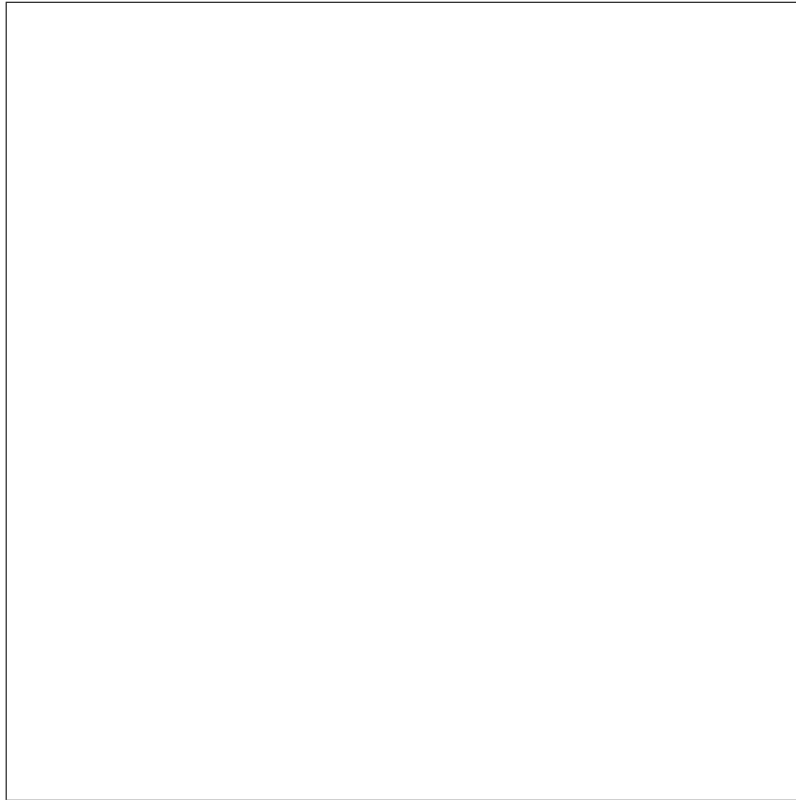
$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

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<sup>1</sup>Next lecture, we will see that  $\mathbb{C} \sqcup \{\infty\}$  can be seen geometrically as a *Riemann sphere*.



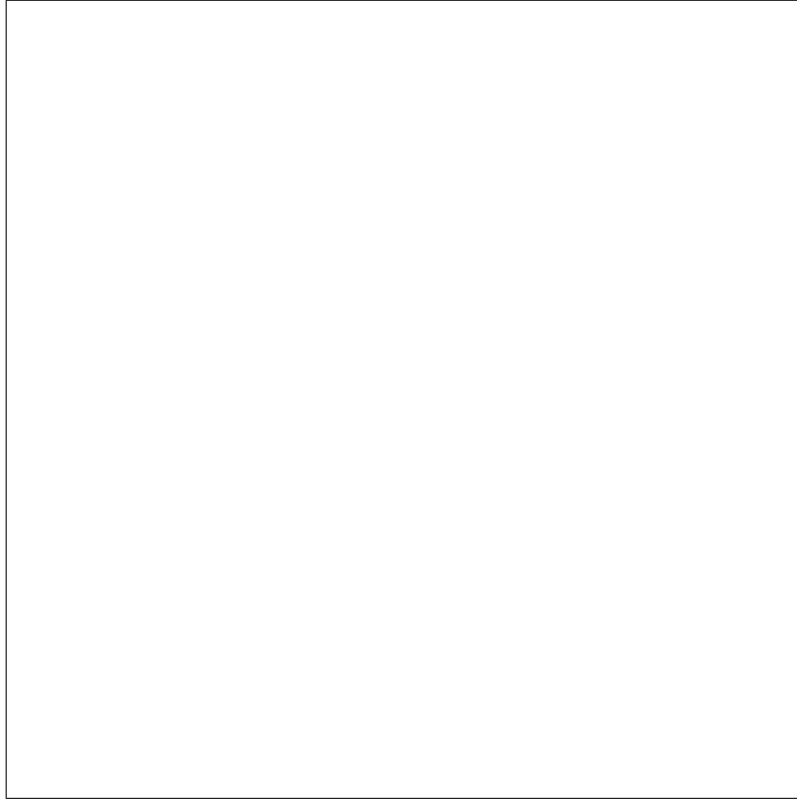
(b) Deduce that the inverse function (inverse *in the sense of composition*) to a fractional linear transformation is a fractional linear transformation, and in particular a function which is holomorphic with at most one pole (hint: use the matrix inverse).



## 5 Bonus questions

### Question 3.

(a) What is the residue of the pole of a fractional linear transformation in terms of  $a, b, c, d$ ?



(b) Show that a function  $f(z)$  which is defined on all of  $\mathbb{C}$  except a single simple pole, and is bounded when  $z \rightarrow \infty$ , is a fractional linear transformation **Hint:** Subtract the singular part, then use Liouville's theorem.

