Worksheet 1: the Gamma function

This worksheet is optional, material from it will not be on the final.

March 11, 2020

1 Introduction

Many functions start their life as a function of the integers, and then turn out to have a remarkably nice extension to the entire real line, and sometimes even the entire complex plane. The example you're most familiar with is the function 2^n. The first time you encounter it, it only makes sense for n a positive integer. But it turns out that n can be replaced by x and even by z (this is an entire function using $2^z := \exp(z \log(2))$, and this extension is unique satisfying certain properties: in this case a sufficient set of properties is: holomorphicity, the requirement that 2^x is real for x real, and the formula $2^{x+y} = 2^x \cdot 2^y$, and the obvious requirement $2^1 = 2$.)

2 The factorial function.

Today we're going to do the same thing with the factorial function $\mathrm{Fact}(n) := n!$, originally defined for n a positive integer.

Here are the properties we would like $\mathrm{Fact}(z)$ to satisfy:

1. $\mathrm{Fact}(0) = 1$
2. $\mathrm{Fact}(z) = z \cdot \mathrm{Fact}(z-1)$.

Notice that these two properties imply by induction that $\mathrm{Fact}(n) = n!$ for n a positive integer. We would like to have the following additional property:

- ~~$\mathrm{Fact}(z)$ is holomorphic.~~

Unfortunately there's a problem with this: the property $\mathrm{Fact}(z) = z \cdot \mathrm{Fact}(z-1)$ can be reformulated as $\mathrm{Fact}(z-1) = \frac{\mathrm{Fact}(z)}{z}$. If the function $\mathrm{Fact}(z)$ is holomorphic at 0 and satisfies $\mathrm{Fact}(0) = 1$, then the function $\frac{\mathrm{Fact}(z)}{z}$ *must* have a simple pole at $z = 0$. So we can write $(-1)! = \infty$. This implies $(-2)! = \frac{\infty}{-2} = \infty$, etc., for all strictly negative integers.

Of course we're no longer afraid of functions with poles (especially simple ones), so we modify

3. Fact(z) is holomorphic everywhere except for poles at $z = -n$, with n a positive integer.

This is sufficient! I will walk you through defining a function that satisfies (1) and (2) and only has poles at negative integers (in particular, it is “meromorphic”: i.e., all singularities at worst poles).

Unfortunately there’s one more difference with the function 2^z : I mentioned that the simple requirements we put on the function 2^z (essentially, $2^1 = 2$ and the multiplicative property of exponents) determines the function uniquely. This is not the case for the function Fact(z): indeed, you can multiply the function Fact(z) by any periodic factor $f(z)$ with period 1 and with $f(0) = 1$ (for example $f(z) = \cos(2\pi z)$, and you will get another function satisfying (1) and (2). If you try hard enough, you can upgrade the requirements to make the function unique. For now, suffice to say that there is unquestionably a “best” (i.e., most natural, most useful, etc.) function that satisfies the conditions Fact should satisfy. Here is its formula:

$$\mathrm{Fact}(z) := \Gamma(z + 1).$$

Wait... I didn’t tell you anything. Now you just have to define the function $\Gamma(z)$ (the Greek letter is a capital Gamma, \Gamma in LaTeX.)

Well, that’s what this worksheet is for.

3 Definition of the Gamma function

The reason the shifted function Γ is used instead of Fact is historic (and related to the zeta function). The analogue of property (1) after shifting becomes $\Gamma(1) = 1$ and the analogue of property (2) becomes $\Gamma(n + 1) = \Gamma(n) \cdot n$: just as natural.

Define the Gamma function as follows:

$$\Gamma(z) := \int_0^\infty x^{z-1} e^{-x} dx.$$

Notice that this is a real integral, but it depends on a complex parameter z.

Question 1. When does this integral converge? Notice that for large x, the inverse exponential term dominates. The only issue is that x^{z-1} might blow up if $\mathrm{Re}(z) < 1$.

Show that the integral defining $\Gamma(z)$ converges for $\mathrm{Re}(z) > 1$ and diverges for $\mathrm{Re}(z) \leq 0$. **Bonus** Show that it in fact converges for $\mathrm{Re}(z) > 0$.

Note that for a fixed x, the function under the integral $x^{z-1}e^{-x}$ is analytic in z (use $x^{z-1} = \exp(\log(x)(z-1))$: here $\log(x)$ is a constant which is the unique real logarithm of a positive number).

We deduce that the integral must also be analytic. If you want to cross all your t's and dot all your i's (i.e., be very logically thorough), proving that one can interchange integration and complex differentiation requires a slightly more careful limit argument.

Question 2. (Optional) Prove carefully that $\Gamma(z)$ is complex differentiable, with derivative given by $\int_0^\infty x^{z-1} \cdot \log(x)e^{-x}dx$ (i.e. with the result of differentiating under the integrand).

Question 3. Compute $\Gamma(1), \Gamma(2), \Gamma(3)$ using ordinary (real) calculus. (Hint: you know what the values of Fact are on nonnegative integers, so you should have a guess for these values!)

Question 4. Prove using integration by parts that $\Gamma(z+1) = z\Gamma(z)$ when $\mathrm{Re}(z) \geq 1$ (if you did the examples in the previous problem, this should be straightforward).

We now have a wonderful candidate for $\Gamma(z)$ (and therefore $\mathrm{Fact}(z)$), except it is only defined (and, notice, holomorphic without poles!) for z with real part ≥ 1. We'd like to now extend it to the rest of the complex plane except the values $-0, -1, -2, \ldots$, where we expect poles. Remember that if an extension exists it is unique, by the extension theorem.

Question 5. (Optional but recommended) (a) If z has real part < 1, define

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)(z+2)\cdots(z+n)},$$

where n is chosen to be large enough that the integral for $\Gamma(z+n+1)$ converges (i.e., $\mathrm{Re}(z+n+1) \geq 1$). Show that this value does not depend on the choice of n, and therefore the function $\Gamma(z)$ is well-defined for all z.

(b) Show that $\mathrm{Fact}(z) := \Gamma(z+1)$ for Γ defined as above z satisfies (1) and (2). (I.e., $\mathrm{Fact}(0) = 1, \mathrm{Fact}(z) = z\,\mathrm{Fact}(z-1)$, and holomorphicity of Fact except at $-1, -2, \ldots$).

(c) Show that $\Gamma(z)$ extended in this way to all of $\mathbb{C}$ is holomorphic except for $z = -1, -2, \ldots$. (Hint: you have already shown that $\Gamma(z)$ is holomorphic for $\mathrm{Re}(z) \geq 1$. Now reduce showing holomorphicity at z to showing holomorphicity at $z+n$ of the function $\frac{\Gamma(z+n+1)}{z(z+1)(z+2)\cdots(z+n)}$, for n sufficiently large.)

(d) Carefully use the holomorphic extension theorem to prove that this function is the *unique* holomorphic extension of the function $\Gamma(z) \mid \mathrm{Re}(z) > 1$ to $\mathbb{C} \setminus \{0, -1, -2, -3, \ldots\}$.