# Worksheet 7: Some exercises (prep for lecture).

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## 1 Introduction

Today I am going to give a few short exercises based on the lectures so far, in preparation for today's lecture.

## 2 Conformal mappings between domains in the plane

Recall that a conformal mapping is a mapping of sets $\Omega \to \Omega'$ which is a continuous, real differentiable bijection and takes angles between curves to angles between curves. (A conformal *map* is a continuous mapping which takes angle to angles, but drops the bijection requirement). We have access to the following results.

**Proposition 1.**

1. *If $f : \Omega \to \Omega'$ is a conformal mapping then the compositional inverse mapping, $f^{-1}$ (as a mapping of sets), is also conformal.*
2. *If $f : \Omega \to \Omega'$ and $g : \Omega' \to \Omega''$ are conformal then $g \circ f : \Omega \to \Omega''$ is also conformal.*

**Theorem 1.** *If $\Omega, \Omega'$ are two open domains in the complex plane $\mathbb{C}$ then $f : \Omega \to \Omega'$ is a conformal mapping if and only if $f$ is holomorphic and a bijection.*

Now conformal mappings give us a way to compare complex domains in a new way.

**Question 1.** *Construct a conformal mapping from the (open) unit disk $D_1$ to the disk $D_2$ of radius 2.*

**Solution.** Just take the map $z \mapsto 2z$.

**Question 2.** *Let $H = \{z \mid \text{Im}(z) > 0$ be the upper half-plane. Show that for any fractional linear function $f = \frac{az+b}{cz+d}$ such that $a, b, c, d \in \mathbb{R}$ are* real *parameters* and the determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is positive *(thanks to Ankit for pointing this out), $f$ determines a conformal mapping from $H$ to $H$.*

**Solution.** First, we show that if $z$ is in the upper half-plane then $f(z)$ is as well, for $f$ as above. Let $z$ be in the upper half-plane, so $\mathrm{Im}(z) = \frac{z-\bar{z}}{2i} > 0$. We compute the imaginary part $\mathrm{Im} f(z) = \frac{f(z)-f(\bar{z})}{2i} = \frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d}$ (the coefficients $a, b, c, d$ do not get conjugated as they are real). Rewrite this as $\frac{\mathrm{Im}((az+d)(c\bar{z}+d))}{|cz+d|^2}$. The denominator is positive and the imaginary part in the numerator is $(ad-bc)\mathrm{Im}z$, which has the same sign as the determinant $ad-bc$.

Now take $g = \frac{az+c}{bz+d}$ be the compositional inverse of $f$. Then $g$ again takes $\mathbb{H}$ to $\mathbb{H}$. So $f, g$ are inverse holomorphic bijections of the upper half-plane to itself, in particular $f$ is a conformal mapping.

**Question 3.** *Construct a conformal mapping from the (open) upper half-plane, $H = \{z \mid \mathrm{Im}(z) > 0$ to the (open) unit disk, $D_1$. Hint: try using a fractional linear transformation (but not with real coefficients in this case).*

**Solution.** Define $f(z) = \frac{z-i}{z+i}$. Note that for $\in \mathbb{H}$, we see $z+i, z-i$ have the same real part and $|\mathrm{Im}(z+i)| > |\mathrm{Im}(z-i)|$. So $|f(z)| < 1$. This means $f$ maps $\mathbb{H}$ to $\mathbb{D}$. Conversely, the inverse $\frac{z+1}{-iz+i}$ can be seen to map $\mathbb{D}$ to $\mathbb{H}$.

**Question 4.** *(a) Construct a conformal mapping $f : S^{vert}_{(-\pi,\pi)} \to \Omega$ from the* vertical *complex strip $S^{vert}_{(-\pi,\pi)}$ given by $\{z \mid -\pi < \mathrm{Re}(z) < \pi\}$ and the complement to the ray of negative numbers, $\Omega := \mathbb{C} \setminus \{-r \mid r \in \mathbb{R}_{\geq 0}$. (Note: $\Omega$ is open, so does not contain $0$).*
*(b) What is the inverse $f^{-1} : \Omega \to S$? Check that, indeed, this is a holomorphic function with nowhere zero derivative.*

**Solution.** Define $f(z) = \exp(iz)$. This maps the vertical strip $S$ to the complement $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$. The inverse is given by $w \mapsto -i\log(w)$, on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$. Its derivative is $1/z$, which is nonzero

**Question 5.** *Construct a conformal mapping $f : S^{vert}_{(0,\pi)} \to H$ from the vertical complex strip of numbers of real part between $0$ and $\pi$ to the upper half-plane $H$.*

**Solution** Take $z \mapsto \exp(iz)$ again.