Official Worksheet 1: Fractional Linear transformations.
(Material from this worksheet may be on exam!)

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1 Introduction

In this short worksheet we will play around with *fractional linear transformations.* These are functions of the form

$$f(z) = \frac{az+b}{cz+d},$$

where a, b, c, d are complex constants, which have certain nice properties.

2 Fractional linear functions

A *rational function* is a function of the form $f(z) = \frac{p(z)}{q(z)}$, where p and q are both polynomials. A *fractional lienar* function is a rational function $f(z) = \frac{p(z)}{q(z)}$ with the following properties:

1. $p = az + b, q = cz + d$ are either constant or linear polynomials in z.
2. $q(z)$ is not zero (otherwise f would be nowhere defined)
3. $f(z)$ is not constant.

Question 1.
(a) Show that the conditions (2), (3) above on f above are equivalent to the condition that the matrix $M := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has nonzero determinant.

Answer. First, assume the matrix M has zero determinant. This means that the rows of M are proportional or zero. If one of the rows is zero then f is either constant and equal to 0 (if denominator) or undefined (if numerator). If the two are proportional with nonzero constant of proportionality λ, then $f(z) = \frac{az+b}{\lambda az+\lambda b} = \frac{1}{\lambda}$, a constant. Conversely, if $az + b$ and $cz + d$ are neither zero nor proportional, then the fraction $\frac{az+b}{cz+d}$ is the quotient of two nonzero, non-proportional functions, hence not a constant function.

(b) Show that the two functions $f(z) = \frac{az+b}{cz+d}$ and $f'(z) = \frac{a'z+b'}{c'z+d'}$ (satisfying (1)-(3) above) are equal if and only if the matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $M' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ are nonzero scalar multiples of each other (i.e., $M' = \lambda M$ for a nonzero complex number λ).

Answer. If the two matrices are proportional by a constant λ, then the two functions are different by the constant $\lambda/\lambda = 1$. Conversely, we need to check that if the two matrices M, M' are not proportional the $f \neq f'$. There are many ways to do this. One is to observe that the matrix "up to proportionality" is determined by the three quotients $a/c, b/d, c/d$, where each is set to ∞ if the denominator is zero (note that both numerator and denominator cannot be zero by our nondegeneracy condition). Now it's enough to show that these quotients can be read off of the function f (as a function, not as a formula with numerator and denominator). And indeed, you can get a/c as the asymptotic value of $f(z)$ as $z \to \infty$ and b/d as the value of $f(z)$ at $z = 0$ (both of these formally set to ∞ if the denominator is zero). Finally, you can get d/c as the pole of f (set to ∞ if f has no poles).

3 As mappings from the plane with ∞ to itself

We see immediately that a fractional linear function is either entire (when $c = 0$ and the denominator is zero) or (when $c \neq 0$), has one pole at $z_0 = -d/c$, the zero of the denominator). Convince yourself that this is a *simple pole*.

Question 2.

(a) Show that, if $c \neq 0$ and $\frac{az+b}{cz+d}$ is a fractional linear transform then $\lim_{z\to\infty} f(z)$ exists and is equal to a/c.

(b) Extend the function f (a function from a domain of definition to $\mathbb{C}$) to a function from the set $\mathbb{C} \sqcup \{\infty\}$, consisting of all complex numbers and the symbol ∞, by setting $f(\infty) = a/c$ (defined to be ∞ if $c = 0$) and $f(z_0) = \infty$ for z_0 the pole (if it exists). Prove that any fractional linear transformation extended in this way becomes a *one-to-one and onto* (i.e., bijective) function from $\mathbb{C} \sqcup \{\infty\}$ to itself.[1]

Answer.(a) is a straightforward limit computation, keeping in mind that $\lim_{z\to\infty}$ means that we take the limit along any sequence z_n with $|z_n| \to \infty$.

[1]Next lecture, we will see that $\mathbb{C} \sqcup \{\infty\}$ can be seen geometrically as a *Riemann sphere.*

4 Compositions

The composition of two fractional linear transformations (as functions from $\mathbb{C} \sqcup \{\infty\}$ to itself) is again a fractional linear transformation, as you can see from the next question.

Question 2. **(a)** Show that the composition (as functions from the set $\mathbb{C} \sqcup \{\infty\}$ to itself) of $f = \frac{az+b}{cz+d}$ and $f' = \frac{a'z+b'}{c'z+d'}$ is the fractional transformation $f'' = \frac{a''z+b''}{c''z+d''}$, where the matrix

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

Answer.(simple calculation that you should do)

(b) Deduce that the inverse function (inverse *in the sense of composition*) to a fractional linear transformation is a fractional linear transformation, and in particular a function which is holomorphic with at most one pole (hint: use the matrix inverse).

Answer. Follows from (a) by noting that if M is a matrix with nonzero determinant then M^{-1} is also a matrix with nonzero determinant. Let g be the function corresponding to M^{-1}. Then $M \circ M^{-1} = I$. Now I corresponds to $\frac{1z+0}{0z+1}$, so $f \circ g = \frac{z}{1} = z$. Similarly, $g \circ f = 1$ since $M^{-1}M = I$.

5 Bonus questions

Question 3.

(a) What is the residue of the pole of a fractional linear transformation in terms of a, b, c, d?

Answer. Since the pole is simple, it is enough to find the leading term of the Lauren series. If f has a pole, we must have $c \neq 0$ (else the denominator is constant). Write $\frac{az+b}{cz+d}$ as

$$f(z) = \frac{\frac{a}{c}z + \frac{b}{c}}{z + \frac{d}{c}} = \frac{L(z)}{z - z_0},$$

for $z_0 = -\frac{d}{c}$ and $L(z) = \frac{a}{c}z + \frac{b}{c}$ the linear numerator. This gives a residue of $\frac{a}{c}z_0 + \frac{b}{c} = \frac{ad}{c^2} + \frac{b}{c}$.

(b) Show that a function $f(z)$ which is defined on all of $\mathbb{C}$ except a single simple pole, and is bounded when $z \to \infty$, is a fractional linear transformation **Hint:** Subtract the singular part, then use Liouville's theorem.

Suppose the pole is at z_0. Then near the pole, $f(z) = \frac{R}{z-z_0} + O(1)$ (here R, a constant, is the residue at z_0 and $O(1)$, representing lower order terms, is a function with no pole at z_0). Set $g(z) := f(z) - \frac{R}{z-z_0}$. Then g has no pole at z_0

by the way we constructed it. And since neither $f(z)$ nor $\frac{R}{z-z_0}$ has poles other than z_0 (this is our assumption on f), we see that $g(z)$ has no poles, i.e. is entire. On the other hand, both $f(z)$ and $\frac{R}{z-z_0}$ is bounded outside a neighborhood of $z - z_0$. This means that g is continuous and bounded outside a compact set, therefore $g(z)$ is bounded. Entire and bounded implies constant, so $g(z) = G$, some other constant. We deduce $f(z) = G + \frac{R}{z-z_0} = \frac{G(z-z_0)+1}{z-z_0}$, a fractional linear transformation.