# Math 185 Homework 6.Due Wednesday 3/4

**Do 5 out of the first six problems in Chapter 3 (Stein and Shakarchi problems III.8.1-III.8.6).**

**III.8.1** Using Euler's formula, $\sin(\pi z) = 0$ if and only if $\exp(\pi i z) = \exp(-\pi i z)$. Dividing by the RHS gives $\exp(2\pi i z) = 1$, which is true if and only if $z \in \mathbb{Z}$ (is an integers). The residue of $\frac{1}{\sin(z)}$ at $n \in \mathbb{Z}$ is

$$\frac{1}{\sin'(\pi n)} = \frac{1}{\cos(\pi n)} = \begin{cases} 1, & n \text{ even} \\ 2, & n \text{ odd} \end{cases}.$$

**III.8.2** Let $f(z) = \frac{1}{1+z^4}$. We can write $f(z) = (z+i)^{-2}(z-i)^{-2}$, and in particular $f$ only has singularities at $\pm i$. Set $g(z) = (z+i)^{-2}$. Then $f(z) = \frac{g(z)}{(z-i)^2}$. Write the Taylor expansion around $i$ of $f(z)$, so $f(z) = \sum a_k \Delta z^k$ for $\Delta z = z - i$. Then if $g(z) = \sum b_k \Delta z^k$, we have $f(z) = \frac{g(z)}{\Delta z}^2 = \sum b_k z^{k-2}$. This means that $a_k = b_{k+1}$. In particular the residue $\text{Res}_{z=i} f(z)$ is $a_{-1} = b_1 = g'(i) = \frac{-2}{(i+i)^3} = \frac{-2}{-8i} = \frac{-i}{4}$.

Now let $I_h(R) = \int_{-R}^{R} \frac{1}{1+z^4}$ be the integral of $f(z)$ along a long (real) horizontal segment $\gamma_h = [-R, R]$. Obviously, $\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} I_h(R)$. Let $I_a(R) = \int_{\gamma_a} f(z)dz$ be the integral of $f$ along the semicircle $\gamma_a(t) = R\exp(it)$ for $t \in [0, \pi]$. Then the union $\gamma_h \cup \gamma_a$ is a simple closed loop, which we call $\gamma$. When $R > 1$, the interior of $\gamma$ contains a single singularity of $f$, namely $i$, so the loop integral can be expressed in terms of a residue[1]. Thus $\int_\gamma f(z)dz = 2\pi i \text{Res}_i f = 2\pi i \frac{-i}{4} = \frac{\pi}{2}$. But notice that by the triangle inequality, $|f(z)| \le \frac{1}{R^4-1}$ for $z$ on the arc $\gamma_a$. This means that $I_a(R) \le \int_0^\pi |f(z)| \cdot |\dot\gamma(z)| dt \le \frac{\pi R}{R^4-1}$. As $R \to \infty$, we have $I_a(R) \to 0$, so

$$\frac{\pi}{2} = \lim_{R\to\infty} \int_\gamma f(z)dz = \lim_{R\to\infty} I_a(R) = \int_{-\infty}^{\infty} \frac{1}{x^4+1}.$$

**Notice: we could have just as well taken the lower semicircle instead of the upper semicircle, as the integral along the arc would go to zero. This would reverse the parametrization of the horizontal component $\gamma_h$, flipping a sign. This is telling us that relevant residues at at $i$ and at $-i$ differ by a sign.**

[1] Note that here the "residue" point of view could equally well be replaced by applying Cauchy residue formula for $g'(i)$ to the function $g(z) = \frac{1}{z+i}$.

**III.8.3**

Let $f(z) = \exp(iz)$. Theb for real $x$, we have $\mathrm{Re}(\exp(ix)) = \cos(x)$ and $\mathrm{Re}(\frac{\exp(ix)}{x^2+a^2}) = \frac{\cos(x)}{x^2+a^2}$. Therefore it is enough to find $\int_{-\infty}^{\infty} \frac{\exp(iz)}{z^2+a^2}$, then take the real part[2]. So define $I_h(R) := \int_{-R}^{R} \frac{\exp(ix)}{x^2+a} dz$, the path integral along the horizontal segment $[-R, R]$. Let $\gamma_a$ be the arc $R\exp(it)$ for $t \in [0, \pi]$, as before. Let $\gamma$ be the composed contour, consisting of traversing $[-R, R]$, then $\gamma_a$, counterclockwise. Let $g(z) = \frac{\exp(iz)}{z+ia}$. Then (assuming $R > 1$), we see that on the interior of the closed loop $\gamma$ the function $g(z)$ is holomorphic, and so by Cauchy we can compute

$$\int_\gamma f(z)dz = \int_\gamma \frac{g(z)}{z-i}dz = 2\pi i g(z) = 2\pi i \frac{\exp(i(ia))}{ia+ia} = \frac{\pi \exp(-a)}{a}.$$

Now we once again observe that for $z \in \gamma_a$, we have $\mathrm{Im}(z) \geq 0$ so $|\exp(iz)| \leq 1$ and $\frac{1}{|z^2+a^2|} \leq |R^2 - a^2|$. Thus the limit

$$\lim_{R\to\infty} |\int_{\gamma_a} f(z)dz| \leq \lim_{R\to\infty} (\frac{R}{R^2-a^2}) = 0.$$

It follows that $\pi\frac{e^{-a}}{a} = \lim_{R\to\infty} \int_\gamma f(z)dz = \lim_{R\to\infty} \int_{\gamma_h} f(z)dz + 0$. Taking real parts doesn't change the answer in this case: $\pi\frac{e^{-a}}{a} = \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+a^2}$.

**III.8.4** This problem is very similar to the last one, except that we take imaginary parts and the value of $g$ at $z_0 = ai$ gets multiplied by a factor of $ai$.

**III.8.5** If $\xi$ is negative, we take a contour integral of $\frac{\exp(-2\pi i x\xi)}{(1+z^2)^2}$ in the positive plane, along a curve consisting of a semicircle of radius $R$ and an interval from $-R$ to $R$. If $\xi$ is negative, we take the integral of the same function along the reflected curve in the positive plane. In the first case, we traverse the interval in the direction $R \to -R$, and pick up a term of the type $\int_{-R}^{R} \frac{e^{-\pi i z\xi}}{(1+z^2)^2}$ (with limit $\int_{-\infty}^{\infty} \frac{e^{-\pi i z\xi}}{(1+z^2)^2}$) in the residue calculation; the semicircle term goes to zero. In the otehr direction we pick up a term of the type $\int_{R}^{-R} \frac{e^{-\pi i z\xi}}{(1+z^2)^2}$, with the parametrization of the interval $[-R, R]$ given by going backwards, which is (in the limit) the negative integral, (visually: we are traversing a contour in the lower half-plane counterclockwise, so $[-R, R]$, which is at its top, must be parametrized against the standard parametrization.) The rest of the problem is a straightforward residue calculation in the corresponding contour. Notice that here taking the opposite contours wouldn't work, as the relevant function would grow exponentially on the arc component!

[2] Alternatively, we can use $\cos(x) = \frac{1}{2}(\exp(ix)+\exp(-ix))$, and then done a contour integral calculation for $\frac{\exp(iz)}{z^2+a^2}$ and for $\frac{\exp(-iz)}{z^2+a^2}$, while taking care that the contours we would use for the two functions are mirror to each other

**III.8.6**

Start with $g(z) = \frac{1}{(z+i)^{n+1}}$. Since the degree in the denominator is $\geq 2$, the integrals along arcs of radius $R$ approaches 0 in the $R \to \infty$ limit, and the contour integral reduces to an integral along the boundary of a semicircle with pole at $i$, as before. We would like to know the integral $\oint \frac{g(z)}{(z-i)^{n+1}}$. Write $\Delta z = z - i$. Then $g(z) = \frac{1}{(\Delta z - 2i)^{n+1}}$ and the relevant residue is the $(\Delta z)^n$-term of the function $\frac{1}{(\Delta z - 2i)^{n+1}}$. We use the geometric formula for the inverse, first rewriting it in terms of $\frac{1}{1-\beta}$ for some $\beta = O(\Delta z)$ a small (in the $\Delta z \to 0$ limit) parameter depending on $\Delta z$. In this case we have

$$\frac{1}{(\Delta z + 2i)^{n+1}} = \frac{1}{(2i)^{n+1}(1-\beta)^{n+1}},$$

for $\beta = \frac{-z}{2i} = \frac{zi}{2}$. The formula now follows from considering the expression $\frac{1}{(1-\beta)^{n+1}} = \sum_{k=0}^{\infty} \binom{n+k}{k} \beta^k$, looking at the $\Delta z^n$ term in this expression, and using the fact that

$$\binom{2n}{n} \cdot 2^{-2n} = (2n!)(2^n n!)^{-1}(2^n n!^{-1}) = \frac{1 \cdot 2 \cdot \dots \cdot (2n-1) \cdot 2n}{2^2 \cdot 4^2 \cdot 6^2 \cdot \dots (2n)^2}.$$