Math 185 Homework 5 solutions and hints.

1. (a) Assume that $f(z)$ is an everywhere holomorphic function which is *periodic* with period π, so that $f(z+\pi) = f(z)$. Show that if f is bounded on the strip $\{a + bi \mid -\pi/2 \leq a \leq \pi/2\}$ then f is a constant.

The boundedness statement is equivalent to saying that there exists a real constant $c > 0$ such that $|f(a+bi)| \leq c$ for $\pi/2 \leq a \leq \pi/2$. Now if $z = a + bi$ is arbitrary then by periodicity, $f(z) = f(z')$ for $z' = a' + bi$, with $a' = (a + \pi/2 \mod \pi) - \pi/2$. Since $0 \leq a \mod \pi \leq \pi$, we see that $-\pi/2 \leq a' \leq \pi/2$, so $|f(z)| = |f(z')| \leq c$.

(b) Show that the sum $f(z) := \sum_{n=-\infty}^{\infty} \frac{1}{(z+\pi\cdot n)^2}$ converges for all z other than integer multiples of π (for which one of the $\frac{1}{z+\pi\cdot n}$ will blow up), and show that it is periodic with period π.

First, assume $z = a + bi$ and $-\pi/2 \leq a \leq \pi/2$. Write $f(z) = 1/(z+0)^2 + \sum_{n\neq 0} \frac{1}{(z+n\pi)^2}$, or equivalently (separating positive and negative n), we write $f(z) = \sum_{n\geq 1} \frac{1}{(z+n\pi)^2} + \sum_{n\leq -1} \frac{1}{(z+n\pi)^2}$. Now since $|a| \leq \pi/2$, the triangle inequality gives $|a + n\pi| \geq (|n| - 1/2)\pi$, and so $|z| \geq |\mathrm{Re}(z)| \geq (|n| - 1/2)\pi$. Flipping the inequality, we see that for a nonzero integer n, we have the inequality $|\frac{1}{z+n\pi}| \leq \frac{1}{((|n|-1/2)\pi)^2} = \frac{4}{(2n-1)^2\pi^2}$. This converges (use the comparison test or the integral test), so so long as the remaining term $\frac{1}{z+0\pi}$ is defined (i.e. $z \neq 0$) we see that the sum defining $f(z)$ converges absolutely.

Now if $z = a + bi$ is any complex number which is not an integer multiple of π, let $z' = (a + \pi/2) \mod \pi - \pi/2$, so that $z' \equiv z \mod \pi$ and $|z'| \leq \pi/2$. Say that $z' = z + k\pi$, for $k \in \mathbb{Z}$. Then each summand in the sum for $f(z)$ is $\frac{1}{(z+n\pi)^2} = \frac{1}{(z'+(n-k)\pi)^2}$, and is also a summand in the sum for $f(z')$. And conversely, each summand $\frac{1}{(z'+n\pi)}$ is equal to the summand $\frac{1}{(z+(n+k)\pi)}$ in the sum for $f(z)$ (note that this is not true if we do not allow n to be negative — why?)

So we've seen that $f(z')$ is defined by an absolutely convergent sum for $z' \neq 0$ and $f(z)$ is defined by a rearrangement of this sum (for $z' \neq 0 \mod \pi$). Therefore the sum defining $f(z')$ converges to the same value as $f(z)$. This gives periodicity and well-definedness.

(c) You may assume $f(z)$ defined above is holomorphic (on the domain $z \neq \pi n$). Show that f is bounded for z satisfying $|\mathrm{Im}(z)| \geq 1$.

Say that $z = a + bi$. Then $f(z) = f(z')$ for $z' = a' + b$ with $|a'| \leq \pi/2$, as before. We are now assuming that $|b| \geq 1$. Now from before, $\sum_{n\neq 0} \frac{1}{(z'+n\pi)}$ is bounded by the constant $c = 2\cdot\sum_{n=0}^{\infty} \frac{4}{(2n-1)^2\pi^2}$ (the 2 is here because each term $\frac{1}{(|n|\pi - \frac{1}{2}\pi)}$ appears twice: once for a negative and once for a positive index). So it remains to bound $\frac{1}{z'}$. But $|z'| \geq \mathrm{Im}(z') = \mathrm{Im}(z) \geq 1$, so we get that $f(z) = f(z') \leq 1 + c$ when $|\mathrm{Im}(z)| \geq 1$.

(d) We will see later (when we study Laurent series) that the poles of $f(z)$ exactly cancel the poles of $\frac{1}{\sin(z)^2}$, so that $f(z) - \frac{1}{\sin(z)^2}$ is everywhere holomorphic (or rather, can be extended to an everywhere holomorphic function). Taking this on faith, show that $f(z) = \frac{1}{\sin(z)^2} + c$ (for some constant c). (Hint: it will be helpful to see that $\frac{1}{\sin(z)^2}$ is also bounded for $|\mathrm{Im}(z)| \geq 1$.)

Suppose $|\mathrm{Im}(z)| \geq 1$. Then $|\sin(z)| = |\frac{e^{iz}+e^{-iz}}{2i}| \geq \frac{1}{2}|(|e^{iz}| - |e^{-iz}|)|$, by the triangle inequality. But $|e^z| = e^{\mathrm{Re}(iz)} = e^{-\mathrm{Im}(z)}$. If $\mathrm{Im}(z) \geq 1$ then $|e^{iz}| \leq e^{-1}$ and $|e^{-iz}| \geq e$. Conversely, if $\mathrm{Im}(z) \leq 1$ then $e^{iz} \geq e$ and $e^{-iz} \leq e^{-1}$. In either of these cases (i.e., if $|\mathrm{Im}(z)| \geq 1$) we have by the triangle inequality $|\sin(z)| \geq \frac{1}{2}(e - e^{-1})$. Squaring and taking reciprocals, $\frac{1}{|\sin(z)|^2} \leq \frac{4}{(e-e^{-1})^2}$, giving the desired bound.

We are given that $f(z) - \frac{1}{\sin(z)^2}$ is a holomorphic (therefore continuous) function. By compactness, it is bounded on the rectangle $\{a+bi \mid |a| \leq \pi/2, |b| \leq 1\}$. We have seen that both $f(z)$ and $\frac{1}{\sin(z)^2}$ is bounded on the strip $\{a+bi \mid |a| \leq \pi/2, |b| \geq 1\}$. Together, these bounds imply that $f(z) - \frac{1}{\sin(z)^2}$ is bounded for $z = a+bi$ in the strip with $|a| \leq \pi/2$. By periodicity, we deduce that $f(z) - \frac{1}{\sin(z)^2}$ is a bounded, everywhere holomorphic function. Applying Liouville's Theorem, we see it is constant.

2. Do the following problem (a-g) from Gamelin.

Exercises for IV.4

1. Evaluate the following integrals, using the Cauchy integral formula:

(a) $\displaystyle\oint_{|z|=2} \frac{z^n}{z-1}\,dz, \quad n \geq 0$

(b) $\displaystyle\oint_{|z|=1} \frac{z^n}{z-2}\,dz, \quad n \geq 0$

(c) $\displaystyle\oint_{|z|=1} \frac{\sin z}{z}\,dz$

(e) $\displaystyle\oint_{|z|=1} \frac{e^z}{z^m}\,dz, \quad -\infty < m < \infty$

(f) $\displaystyle\int_{|z-1-i|=5/4} \frac{\mathrm{Log}\, z}{(z-1)^2}\,dz$

(g) $\displaystyle\oint_{|z|=1} \frac{dz}{z^2(z^2-4)e^z}$

Apply Cauchy! Answers: (a) $2\pi i$, (b) 0, (c) 0, (e) For $m \geq 1$, we have $\frac{2\pi i}{(m-1)!}$ and for $m \leq 0$ the integral is zero (the integrand is holomorphic in the interior). (f) $2\pi i$ (the function log is defined on this contour, as it contains no real nubmers ≤ 0 and the contour contains $z_0 = 1$ in its interior), so we get $2\pi i \log'(1) = 2\pi i$. (g) This is the derivative at zero of the function $f(z) = \frac{1}{(z^2-4)e^z}$, which is holomorphic in the unit disk. This is $\frac{1}{4}2\pi i = \frac{\pi i}{2}$.

3. Stein-Shakarchi II.6.8 (the Cauchy Inequalities are Corollary 4.3 on page 48).

Since holomorphic functions have all continuous derivatives, $f^{(n)}(x)$ is bounded on a bounded interval, and so we can make the inqeuality true for $|x| \leq 3/2$. It is therefore enough to find a constant A_n that holds for $|x| \geq 3/2$. Let Ω be the strip $\text{Im}(z) < 1$. Let $x \in \mathbb{R}$. Let $D_{1/2}(x)$ be the disk around x of radius $1/2$ and $C_{1/2}$ the circle around x of radius $1/2$. We have $f^{(n)}(x) = c \int_{C_{1/2}(x)} f(z)dz$, where $c = \frac{1}{n! \cdot 2\pi i}$ is a constant. The Cauchy inequality[1] tells us that $|f^{(n)}(x)| \leq c_n \cdot \text{Max}_{z \in C_{1/2}(x)} |f(z)|$ (for $c_n = n! \cdot \left(\frac{1}{2}\right)^{-n}$). Since $|x| \geq 1/2$, if $z \in C_{1/2}(x)$ the triangle inequality gives $|x| - 1/2 \leq |z| \leq |x| + 1/2$. The condition on f then implies that $|f(x)| \leq A(1+|z|)^\eta \leq \min(A \cdot (1+|x| \pm 1/2)^\eta)$ (plus or minus depending on whether η is positive or negative). Now $1+|x| \pm 1/2 \in \leq 2(1+|x|)$, so $f(z)$ is at most a factor of 2^η more than $A(1+|x|)^\eta$.

We have

$$\int_{C_{1/2}} \frac{f(z)}{(z-x)^{n+1}} dz = \int_{\theta=0}^{2\pi} \frac{f\left(x + \frac{1}{2}\exp(i\theta)\right)}{\cdot} \frac{i}{2} \exp(i\theta),$$

using that $\frac{d}{d\theta} \exp(i\theta) = i \exp(\theta)$. Now $\frac{1}{2}\exp(\theta)$, and therefore $\text{Max}_{z \in C_{1/2}(x)} |f(z)| \leq 2^\eta (1+|x|)^\eta$, and we are done by the Cauchy inequality.

4. Stein-Shakarchi II.6.9

Let $\phi_0(z) = \phi(z+z_0) - z_0$. Let $\Omega_0 = \Omega - z_0$ (the shift of Ω that moves z_0 to 0). Then ϕ_0 takes Ω_0 to Ω_0 and we have $\phi(0) = 0$ and $\phi_0'(0) = \phi'(z_0) = 1$. Therefore we may assume without loss of generality that $\phi = \phi_0$ and $z_0 = 0$. Write $\phi(z) = \sum a_j z^j$. Since We have $a_0 = \phi(0) = 0, a_1 = \phi'(0) = 1$. Assume $\phi(z)$ is not linear. Then there is some minimal $n \geq 1$ such that $a_n \neq 0$. Define $F_k(z) := \phi \circ \cdots \circ \phi(z)$, the k-fold self-composition of ϕ. By induction on k, we can show that $F_k(z) = z + k a_n z^n + O(z^{n+1})$. Indeed: assume this is true about F_k. Then using that $F(z) = O(z)$, we have $F_{k+1} = F_k(\phi(z)) = \phi(z) + k a_n \phi(z)^n + O(\phi(z)^{n+1}) = (z + a_n z^n + O(z^{n+1})) + k a_n (z + a_n z^n)^n + O(z)^{n+1} = z + (k+1) a_n z^n + O(z^{n+1})$. On the other hand, $\phi(\Omega) \subset \Omega$, so $F_n(\Omega) \subset \Omega$. Now choose a circle C_r around 0 such that C_r and its interior is contained in Ω. Then, for some real constant $A > 0$ (depending on n, r), we must have $\frac{k a_n}{n!} = F_k^{(n)} \leq A\ddot{M}ax_{z \in C}(|F_k(z)|)$. Now recall that Ω is bounded. Choose R such that $z \in \Omega$ implies $|z| < R$. Now since $\phi(\Omega) \subset \Omega$, we must have $\phi \circ \phi(\Omega) \subset \Omega, \ldots, F_k(\Omega) \subset \Omega$. Thus $\ddot{M}ax_{z \in C_r} F_n(z) \leq R$

[1] The Cauchy inequality generalizes the observation that $|\frac{f(\exp(t))}{\exp(it)^{n+1}} \dot{\exp}(it)| = |f(\exp(it))|$, so $|\frac{1}{\pi n!} f^{(n)}(0)| \leq \int_0^{2\pi} |f(\exp(it))|$. If C is a more general circle with radius r, centered around z_0, then the bound on the right gets rescaled by a power of the radius, $\frac{1}{\pi n!}|f^{(n)}(z_0)| \leq \frac{1}{r^n} \int_{z \in C} |f(z)| d\theta$, where the integral is parametrize by angle. The integral can be further bounded $\int_{z \in C} |f(z)| d\theta \leq 2\pi \cdot ||f_C||$ where f is the maximum of f on C.

(regardless of what k is). So $\frac{k|a_n|}{n!} \leq AR$, or in other words $k \leq \frac{n!AR}{|a_n|}$ (the right hand side is defined since we assumed $a_n \neq 0$). Thus the finite constant $\frac{n!AR}{|a_n|}$ is greater than any positive integer, which is clearly absurd. This contradicts our assumption that ϕ is not linear.

5. (Bonus) instead of one of the above exercises you can do problem II.7.1. **(a)** Let $z = \exp\left(\frac{2\pi ik}{2^N}\right)$. Notice that such points are dense in the unit circle. $z^{2^n} = 1$ for $n \geq N$. So for real $0 < r < 1$, we have $f(rz) = \sum_{n<N}(rz)^{2^n} + \sum_{n \geq N}(r)^{2^n}$. The first sum is bounded by N (by the triangle inequality, as $|rz^{2^n}| \leq 1$). But as $r \to 1$, the second sum approaches $\sum_{n \geq N} 1 = \infty$. Thus at a dense set of points z of C_1, the function $f(z)$ is unbounded in a neighborhood of z (in the open disk), and this implies that there is no point of C_1 near which $f(z)$ is bounded, and f cannot be continued holomorphically (or even continuously) to any point of the circle.

(b) By an absolute convergence argument, the function f is defined and continuous on the closure of the unit disk. Define $g(z) = f^{(a)}(z)$, the ath derivative of f, with $a > \alpha$. Then asymptotically the kth coefficient of the Taylor series of $g(z) = \sum_{k=0}^{\infty} b_k z^k$ is $b_k = O(2^{na-n\alpha}) \cdot z^{2^n - a}$, with constant of proportionality equal to 1. In particular, these coefficients are positive and go to ∞. By a similar argument to the above, if $z = \exp\left(\frac{2\pi ik}{2^N}\right)$ and $0 < r < 1$ is real then

$$g(rz) = \sum_{n \leq N} b_k (rz)^k + \sum_{n \geq N} b_k \cdot r^k \cdot z^{-a}$$

(since $z^{2^N - a} = z^{-a}$). The first sum is bounded independently of r and the second sum goes to $\infty \cdot z^{-a}$ for $r \to 1$. Thus $g(z) = f^{(a)}(z)$ is nowhere bounded on the circle C_1, and cannot be extended continuously to any point of C_1. But if $f(z)$ had an analytic continuation at some point of the circle, all its derivatives would again be analytic, and in particular continuous. Contradiction.