Math 185 Homework 5.Due Wednesday 2/25

Exercise II.6.n denotes exercises Stein and Shakarchi, Chapter 2, section 6. II.7.n is similar for "problem n".

1. (a) Assume that $f(z)$ is an everywhere holomorphic function which is *periodic* with period π, so that $f(z+\pi) = f(z)$. Show that if f is bounded on the strip $\{a + bi \mid -\pi/2 \leq a \leq \pi/2\}$ then f is a constant.

(b) Show that the sum $f(z) := \sum_{n=-\infty}^{\infty} \frac{1}{(z+\pi \cdot n)^2}$ converges for all z other than integer multiples of π (for which one of the $\frac{1}{z+\pi \cdot n}$ will blow up), and show that it is periodic with period π.

(c) You may assume $f(z)$ defined above is holomorphic (on the domain $z \neq \pi n$). Show that f is bounded for z satisfying $|\mathrm{Im}(z)| \geq 1$.

(d) We will see later (when we study Laurent series) that the poles of $f(z)$ exactly cancel the poles of $\frac{1}{\sin(z)^2}$, so that $f(z) - \frac{1}{\sin(z)^2}$ is everywhere holomorphic (or rather, can be extended to an everywhere holomorphic function). Taking this on faith, show that $f(z) = \frac{1}{\sin(z)^2} + c$ (for some constant c). (Hint: it will be helpful to see that $\frac{1}{\sin(z)^2}$ is also bounded for $|\mathrm{Im}(z)| \geq 1$.)

2. Do the following problem (a-g) from Gamelin.

Exercises for IV.4

1. Evaluate the following integrals, using the Cauchy integral formula:

(a) $\displaystyle\oint_{|z|=2} \frac{z^n}{z-1}\, dz, \quad n \geq 0$

(b) $\displaystyle\oint_{|z|=1} \frac{z^n}{z-2}\, dz, \quad n \geq 0$

(c) $\displaystyle\oint_{|z|=1} \frac{\sin z}{z}\, dz$

(e) $\displaystyle\oint_{|z|=1} \frac{e^z}{z^m}\, dz, \quad -\infty < m < \infty$

(f) $\displaystyle\int_{|z-1-i|=5/4} \frac{\operatorname{Log} z}{(z-1)^2}\, dz$

(g) $\displaystyle\oint_{|z|=1} \frac{dz}{z^2(z^2-4)e^z}$

3. Stein-Shakarchi II.6.8 (the Cauchy Inequalities are Corollary 4.3 on page 48).

4. Stein-Shakarchi II.6.9

5. (Bonus) instead of one of the above exercises you can do problem II.7.1.