Math 185 Homework 4 solutions and hints.

Do two of the following three problems. If you do all three, indicate which two you want graded.

1. Let λ be a complex number and let $\Omega = \mathbb{C} \setminus \lambda \cdot \mathbb{R}_{\geq 0}$ be the complement in $\mathbb{C}$ to all real positive multiples of λ.
(a) Show that the function $z \mapsto z^3$ has a continuous inverse function, called $\sqrt[3]{z}$, on Ω. (Hint: polar coordinates might help). Prove that there are exactly three different such continuous functions. Deduce that there is no continuous extension of $\sqrt[3]{z}$ on all of $\mathbb{C} \setminus \{0\}$ (hint: if such a function existed, it would extend one of the three functions you defined. Now try to check continuity.)

Remember the log function, $\log(z) = \ln|z| + i(\arg(-z) - \pi)$: it is set up to satisfy $e^{\log(z)} = z$ and to be continuous for $z \in \mathbb{C} \setminus -\mathbb{R}_{\geq}0$ (the complement to the closed negative ray). Let $\alpha_1, \alpha_2, \alpha_3$ be the three cube roots of λ (we know there are three since the argument can be $\arg(\lambda) + k\frac{2\pi}{3}$ for any k), and define a function

$$f_i(z) = \exp\left(\frac{\log\left(\frac{z}{-lambda}\right)}{3}\right) \cdot \alpha_i.$$

This function satisfies $f(z)^3 = z$ and it is continuous for $z \in \mathbb{C} \setminus (-\lambda)(-\mathbb{R}_{\geq 0})$, which is precisely Ω (the complement to the λ-ray).

Conversely, suppose that $f(z)$ is a continuous function on Ω which satisfies $f(z)^3 = z$. Let f_1 be one of the three functions defined above. Then $f(z)^3 = f_1(z)^3$, so $f(z)/f_1(z)$ takes values in cube roots of 1, i.e. in $\{1, \zeta_3, \zeta_3^2\}$ for ζ_3 the primitive cube root of unity. By continuity, $f(z)/f_1(z)$ cannot jump from one value to another along any path, and so since Ω is connected, $f(z)/f_1(z)$ must be constant, equal to one of $1, \zeta_3, \zeta_3^2$. Therefore there are precisely three different cube root functions, and they must coincide with the f_1, f_2, f_3 above.

(b) Show that $\sqrt[3]{z}$ (for any of the functions you defined on Ω) is holomorphic.

We defined f_1, f_2, f_3 as a composition of holomorphic functions, therefore holomorphic.

2. (a) Assume $\gamma : [0,T] \to C_1$ is a path all of whose values are on the unit circle, $C_1 \subset \mathbb{C}$. Define $U(\gamma)$, the *unwinding* of γ, to be the path $[0,T] \to i\mathbb{R}$ to the imaginary axis given by $t \mapsto \int_0^t \frac{\dot\gamma(u)}{\gamma(u)} du$. Show that the composition $\exp \circ U(\gamma) = \frac{\gamma}{\gamma(0)}$ as functions $[0,T] \to \mathbb{C}$ (for starters, notice that $\exp \circ U(\gamma)$ has values on the unit circle since $U(\gamma)$ has values on the imaginary line).

See that $\exp(i \int_0^t \dot\gamma(u)\gamma(u)) = \frac{\gamma(t)}{\gamma(0)}$ by differentiating both sides.

(b) Deduce that if $\gamma : [0,T] \to C_1$ is a loop with $\gamma(0) = \gamma(T)$, then $U\gamma(T) = 2\pi i k$ for some $k \in \mathbb{Z}$. This number k is called the *winding number* of the loop γ and denoted $W(\gamma)$. Compute the winding number of the loop $\gamma(t) = \exp(it)$ for $t \in [0, 2\pi]$.

We have $\exp(U\gamma(T)) = \frac{\gamma(T)}{\gamma(0)} = 1$, since γ is a loop. Therefore $U\gamma(T) = 2\pi i k$.

(c) Show that if $\gamma_s(t)$ is a family of loops which is varying continuously, i.e. it is a continuous function in both $t \in [0,T]$ and $s \in [0,1]$ and $\gamma_s(0) = \gamma_s(T)$ for all $s \in [0,1]$, then the winding numbers $W(\gamma)$ are the same for all s. Hint: you may use that any finite integral $\int_0^{t_0} f_s(t)dt$ of a continuously varying family of functions depends continuously on s.

If $\gamma_s(t)$ is a family of loops that is continuous in s and in t, then $U\gamma_s(T)$ will be a continuous function of s. On the other hand it will only take values which are integer multiples of 2π, and so must be constant. (For a more rigorous proof: look at $\frac{\gamma_s(T)}{i}$. This is a real-valued function of $s \in [0,1]$. If it is not constant, it will take two different values. By the intermediate value theorem, if it takes two different values, it must take all intermediate values, some of which will not be multiples of 2π. Therefore it must be constant.

The upshot of this problem is that the winding number does not change under homotopies.

3. II.5 from Stein-Shakarchi (chapter 2 section 6).

Follow the hint!