Math 185 Homework 4.Due Wednesday 2/19

Do two of the following three problems. If you do all three, indicate which two you want graded.

1. Let λ be a complex number and let $\Omega = \mathbb{C} \setminus \lambda \cdot \mathbb{R}_{\geq 0}$ be the complement in $\mathbb{C}$ to all real positive multiples of λ.
(a) Show that the function $z \mapsto z^3$ has a continuous inverse function, called $\sqrt[3]{z}$, on Ω. (Hint: polar coordinates might help). Prove that there are exactly three different such continuous functions. Deduce that there is no continuous extension of $\sqrt[3]{z}$ on all of $\mathbb{C} \setminus \{0\}$ (hint: if such a function existed, it would extend one of the three functions you defined. Now try to check continuity.)
(b) Show that $\sqrt[3]{z}$ (for any of the functions you defined on Ω) is holomorphic.

2. (a) Assume $\gamma : [0, T] \to C_1$ is a path all of whose values are on the unit circle, $C_1 \subset \mathbb{C}$. Define $U(\gamma)$, the *unwinding* of γ, to be the path $[0, T] \to i\mathbb{R}$ to the imaginary axis given by $t \mapsto \int_0^t \frac{\dot{\gamma}(u)}{\gamma(u)} du$. Show that the composition $\exp \circ U(\gamma) = \frac{\gamma}{\gamma(0)}$ as functions $[0, T] \to \mathbb{C}$ (for starters, notice that $\exp \circ U(\gamma)$ has values on the unit circle since $U(\gamma)$ has values on the imaginary line).
(b) Deduce that if $\gamma : [0, T] \to C_1$ is a loop with $\gamma(0) = \gamma(T)$, then $U\gamma(T) = 2\pi i k$ for some $k \in \mathbb{Z}$. This number k is called the *winding number* of the loop γ and denoted $W(\gamma)$. Compute the winding number of the loop $\gamma(t) = \exp(it)$ for $t \in [0, 2\pi]$.
(c) Show that if $\gamma_s(t)$ is a family of loops which is varying continuously, i.e. it is a continuous function in both $t \in [0, T]$ and $s \in [0, 1]$ and $\gamma_s(0) = \gamma_s(T)$ for all $s \in [0, 1]$, then the winding numbers $W(\gamma)$ are the same for all s. Hint: you may use that any finite integral $\int_0^{t_0} f_s(t)dt$ of a continuously varying family of functions depends continuously on s.

The upshot of this problem is that the winding number does not change under homotopies.

3. II.5 from Stein-Shakarchi (chapter 2 section 6).