Math 185 Homework 3 select solutions and hints.

This homework set is taken from Stein and Shakarchi, Chapter I.4 and II.6 (exercises for all of chapter I and II). Exercise I.n denotes exercise n of chapter I.4, and similarly for II.n.

1. I.13

(a) Assume $Re(f)$ is constant, equal to a. Then both $\frac{\partial f}{\partial x} = f'$ and $\frac{\partial f}{\partial y} = if'$ have real part 0, so $f' = 0$, and f must be constant.

(b) similar to (a).

(c) If $|f| = 0$, we're done. So assume $|f| = r$ for $r \neq 0$.

Option 1: It's enough to show f is constant in a neighborhood of each $z_0 \in \mathbb{C}$ (this would for example imply $f' = 0$ everywhere and f is constant). Let $g(z) = \frac{f(z)}{f(z_0)}$. Then in some neighborhood of z_0, the function g does not take negative real values so $\log(g)$ is defined and holomorphic and satisfies the condition of part (a).

Option 2: Use the polar form of Cauchy in the following form: If $f(z) = r(z)\exp(i\theta(z))$ with (r, θ) the radial and angular part, then (assuming $r(z) \neq 0$) the Cauchy equation is equivalent to $\frac{\partial \theta}{\partial x} = \frac{1}{|z|}\partial r \partial y$ and $\frac{\partial r}{\partial x} = |z| \cdot -\frac{\partial \theta}{\partial y}$.

2. Do **ONE OF I.20, I.21** If you do both, indicate which one you want graded (I recommend trying to do both for fun.) *The notation* $a_n \sim b_n$ *is "asymptotic equality", i.e. the statement that the limit of the quotients* $\lim \frac{b_n}{a_n} = 1$.

I.20 $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$. By an inductive argument, $\left(\frac{1}{1-z}\right)^m = \sum_{n=0}^{\infty} \binom{n+m-1}{m-1} z^n$. Use $\binom{n+m-1}{m-1} = \frac{n \cdots n-m+2}{(m-1)!}$ for the asymptotic.

I.21 $\frac{z}{1-z} = \sum_{k \geq 1} z^k$, and

$$\frac{z^{2^n}}{1 - z^{2^{n+1}}} = \sum_{k \text{ is divisible by } 2^n \text{ but not by } 2^{n+1}} z^k.$$

Now use absolute convergence to rearrange the second sum and see that each z^k appears exactly one in it.

3. I.24, I.25 Don't worry about being rigorous for these, treat them as calculus problems about one-dimensional integrals. *For I.25: n can be negative! You are not allowed to use the antiderivative trick for this computation (though you can use it to check your answer).* No solution: calculus (for I.25, the tricky integrals are to show the usefulness of later contour integral theorems).

4. II.1 Let $f(z) = e^{-z^2}$. Since $f(z) = \sum_{n\geq 0} \frac{z^{2n}}{n!}$ has an everywhere convergent power series, it has an everywhere convergent antiderivative $F(z) = \sum_{n\geq 0} z^{2n+1} n! \cdot (2n+1)$. Therefore its integral around any loop is equal to zero. Now let γ be the contour in figure 14 (in the problem). Since it is a loop, the path integral $\int_\gamma f(z)dz = 0$. This integral is $0 = I_h + I_d + I_c$, where I_h is the integral along the horizontal stretch, I_d is the integral along the diagonal stretch from 0 to $R\zeta_8 = R \cdot e^{\frac{\pi i}{4}}$ and I_c is the integral along the chord. Using the formula $\int_\gamma f(z)dz = \int_{t=0}^T f(\gamma(t)) \cdot \dot\gamma(t)dt$, we have

$$I_h = \int_{t=0}^R e^{-t^2} dt,$$

$$I_d = \int_{t=0}^R e^{-(\zeta_8 t)^2} \cdot (-\zeta_8)dt,$$

$$I_a = \int_{\theta=0}^{\pi/4} e^{-(Re^{i\theta})^2} \cdot iRe^{i\theta} d\theta.$$

All three of I_h, I_d, I_a are functions of R. Let $I_h(\infty) = \lim_{R\to\infty} I_h(R)$, and similarly with $I_d(\infty), I_a(\infty)$. Observe that

$$\frac{I_d(\infty)}{-\zeta_8} = \int_0^\infty e^{-(\zeta_8 t)^2} dt = \int_{t=0}^\infty e^{-it^2} dt = \int_0^\infty \cos(t^2) - i\sin(t^2)dt.$$

Thus the real and imaginary parts of $\frac{I_d(\infty)}{-\zeta_8}$ precisely give the desired integrals.

The key idea now is that, by taking the $R \to \infty$ limit of the equation $I_h(R) + I_d(R) + I_a(R) = 0$ we get $I_h(\infty) = -I_d(\infty) - I_a(\infty)$. The result follows from the following lemma.

Lemma 1. *$I_h(\infty) = \frac{1}{2}\sqrt{\pi}$ and $I_a(\infty) = 0$.*

Proof. We are given $\int_{-\infty}^\infty e^{-t^2} dt = \sqrt{\pi}$. Since e^{-t^2} is an even function, it follows that $I_h(\infty) = \int_0^\infty e^{-t^2} dt = \frac{1}{2}\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$.

Next we compute $I_a(\infty)$. To show this is zero, it is enough to show that $I_a(R)$ is bounded by a function of R that goes to zero. By the triangle inequality $I_a = \int_{\theta=0}^{\pi/4} e^{-(Re^{i\theta})^2} \cdot iRe^{i\theta} d\theta$ is bounded by

$$I_a^{abs} := \int_{\theta=0}^{\pi/4} |e^{-(Re^{i\theta})^2} \cdot iRe^{i\theta}| d\theta,$$

the integral of the absolute value of the integrand. Compute each term individually: $|e^{-(Re^{i\theta})^2}| = |e^{-R^2e^{2i\theta}}| = e^{\mathrm{Re}(-(R^2e^{2i\theta}))} = e^{-R^2\cos(2\theta)}$, and $|iRe^{i\theta}| = R$. Thus

$$I_a^{abs}(R) = \int_0^{\pi/4} e^{-R^2\cos(2\theta)} \cdot R d\theta = R\int_0^{\pi/4} e^{-R^2\cos(2\theta)} d\theta.$$

This is a real integral. Substituting

$$\phi = \pi/2 - 2\theta$$

and using $\cos(\pi/2 - \phi) = \sin(\phi)$, we write

$$I_a^{abs}(R) = \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin(\phi)} d\phi.$$

The intuition is to now observe that for small ϕ, we have $e^{-R^2\sin(\phi)} \approx e^{-R^2\phi}$, and this integral is bounded by $\approx \frac{1}{R^2}\int_0^\infty e^{-x^2}$, and that for ϕ approaching $\pi/2$ the value $e^{-R^2\phi}$ goes to zero much faster than $1/R$[1] One way to make this observation rigorous is to use that for $0 \le \phi \le \pi/2$, the graph of the sin function is convex, so it is bounded below by the line from $(0,0)$ to $(\frac{\pi}{2}, 1)$; in other words, for $\phi \in [0, \pi/2]$ we have $\sin(\phi) \ge \frac{2}{\pi}\phi$. This implies that our integrand $e^{-R^2\sin(\phi)} \le e^{-R^2\frac{2}{\pi}\phi}$. We deduce

$$I_a^{abs}(R) \le \frac{R}{2}\int_0^{\pi/2} e^{-R^2\frac{2}{\pi}\phi^2} d\phi \le$$

$$\frac{R}{2}\frac{\pi}{2R^2}\int_0^{\frac{2R^2}{\pi}} e^{-R^2} \le$$

$$\frac{\pi}{4R}\int_0^\infty e^{-R^2} = \frac{\pi^{3/2}}{8R}.$$

Thus in the $R \to \infty$ limit we have for $c = \frac{\pi^{3/2}}{8}$ a constant that $|I_a(R)| \le I_a^{abs}(R) \le c/R$, and so $I_a(\infty) = 0$. □

Since $I_a(\infty) = 0$, the vanishing of the contour integral implies that $I_d = -I_h = -\frac{\sqrt{\pi}}{2}$.

We now conclude by computing

$$\int_0^\infty \cos(t^2) - i\int_0^\infty \sin(t^2) = \frac{I_d}{-\zeta_8} = I_d \cdot (-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i) = \frac{\sqrt{\pi}}{2} - i\frac{\sqrt{\pi}}{2}.$$

[1] alternatively, one can do a substitution $u = \sin(2\theta), dt = \frac{du}{\sqrt{1-u^2}}$.