Math 185 Homework 2 Selected solutions/sketches/hints.(Due Wednesday 2/5)

Do *4 out of 6* of the following groups of exercises. You are encouraged to try more, but if you do please indicate which ones you do and do not want graded.

This homework set is taken from Stein and Shakarchi, Chapter I.4 (exercises for all of chapter I). Exercise I.n denotes exercise n of chapter I.4

1. I.1, I.3 — for building intuition on complex numbers and algebra

I.1, a. z such that $|z - z_1| = |z - z_2|$ for z_1, z_2 fixed: this is the set of points in the plane equidistant from two points in the plane. If the points are equal, this is all of $\mathbb{C}$; otehrwise, it is the line which perpendicularly bisects the segment $\overline{z_1, z_2}$ (here the "bar" means segment, not conjugate).

b. $1/z = \bar{z}$. We assume $z \neq 0$ (else $1/z$ undefined). Then since $1/z = \frac{\bar{z}}{|z|^2}$, the condition is equivalent to $|z|^2 = 1$, which gives the unit circle.

c. $\mathrm{Re}(z) = 3$: the vertical line with x-intercept 3.

d. $\mathrm{Re}(z) \geq c$, respectively, $\mathrm{Re}(z) \geq c$: the closed, resp., open, half-plane to the right of the line with x-intercept c.

e. $\mathrm{Re}(az + b) > 0$ for $a, b \in \mathbb{C}$: if $a = 0$, this is everything or nothing (depending on the real part of b). Otherwise, this is the plane obtained from the (open) right half plane by subtracting b and dividing by a, which is bounded by the line
$fraciz - \mathrm{Re}(b)a$ (this line has angle from the horizontal with argument $-\arg(a)$ and distance $\frac{|\mathrm{Re}(b)|}{|a|}$).

f. $|z| = \mathrm{Re}(z) + 1$. This is a parabola rotated by ninety degrees. Squaring: both sides of $\sqrt{x^2 + y^2} = x + 1$ gives $x^2 + y^2 = x^2 + 2x + 1$, so $y^2 = 2x + 1$, or $x = \frac{y^2 - 1}{2}$.

g. $\mathrm{Im}(z) = c$: this is the horizontal line with y-intercept c.

I.3: Set (in polar coordinates) $z = re^{i\alpha}$. We're given $z^n = se^{i\phi}$, giving $s = r^n$ so $r = \sqrt[n]{s}$. And $n\alpha \equiv \phi \mod 2\pi$, so $\alpha = \frac{\phi}{n} + \frac{2\pi k}{n} \mod 2\pi$, for $k \in \mathbb{Z}$. Since this only depends on $k \mod n$, there are n possible values: $z = \sqrt[n]{s} \cdot e^{i \cdot \frac{\phi + 2\pi k}{n}}$ for $k = 0, \ldots, n-1$.

2. I.6 (also consider doing I.5, but don't turn it in) — for building intuition on the notion of connected and compact sets

(a) *Check C_z open:* let $z_0 \in C_z$, so it is connected to z by a path γ. Since Ω is open, there is a disk $D_\epsilon(z_0)$ contained in Ω for some ϵ. Each point z' in $D_\epsilon(z_0)$ is connected to z_0 by the line segment $L = \overline{z_0, z'}$ and forming a path consisting of γ followed by L gives a path from z to z'. So $D_\epsilon(z_0)$ is in C_z, hence C_z open. The rest is a straightforward series of arguments in concatenating and reversing paths.

Note: two paths $\gamma : [0, T] \to \mathbb{C}$ and $\gamma' : [0, T'] \to \mathbb{C}$ can be *concatenated* to a function $\gamma \sqcup \gamma' : [0, T + T'] \to \mathbb{C}$ defined by the formula

$$\gamma \sqcup \gamma'(t) = \begin{cases} \gamma(t) & t < T \\ \gamma(t - T) & t \geq T. \end{cases}.$$

The result is a path if and only if $\gamma(1) = \gamma'(0)$. Note: the concatenated path is sometimes called $\gamma' \circ \gamma$ by topologists – the order changes because the paths compose backwards "like functions".

b. Define $\mathbb{Q} + \mathbb{Q}i$ (the set of "Gaussian rational number") to be the set of numbers of the form $a + bi \mid a, b \in \mathbb{Q}$. Since the rational numbers are countable and the product of two countable sets is countable, $\mathbb{Q} + \mathbb{Q}i$ is countable. The idea of the proof now is as follows:

1. Each connected component of Ω contains a number in $\mathbb{Q} + \mathbb{Q}i$ (in fact, infinitely many such), because it is open and $\mathbb{Q} + \mathbb{Q}i$ is dense.

2. Each number of the form $\mathbb{Q} + \mathbb{Q}i$ is contained in at most one connected component of Ω.

3. This implies that the set of connected components of Ω is "smaller than" the set $\mathbb{Q} + \mathbb{Q}i$ (it can be defined as the quotient of a subset of $\mathbb{Q} + \mathbb{Q}i$ by an equivalence relation), and is thus countable.

Following is a more rigorous proof.

Assume $z \in \Omega$. The connected component Ω_z is open, therefore contains all points in some disk $D_\epsilon(z)$, therefore contains all $a + bi \mid |a - \mathrm{Re}(z)| < \epsilon$ and $|b - \mathrm{Im}(z)| < \epsilon$. Since any real interval contains a rational number, it is possible to choose rational a, b which satisfy this property, so $a + bi \in D_\epsilon(z) \subset \Omega_z$, and thus every connected component contains one of this countable set of points.

Since each $a + bi$ is contained in a unique connected componenent, enumerating the set of values $a + bi \in \mathbb{Q} + \mathbb{Q}i$ which are in Ω implies an enumeration of connected components of Ω, possibly with repetition. Dropping repetitions, we get an enumeration of connected components of Ω, hence it is countable.

c. Assume $K \subset \mathbb{C}$ is a compact (equivalently, closed and bounded) set. Let $U = \mathbb{C} \setminus K$ (open since K is closed). Assume K is bounded by R, i.e., has no points of absolute value $> R$. Let $A = \mathbb{C} \setminus \bar{D}_R$ (the complement to a disk is called an "annulus"; this is the complement to a closed disk, thus A is an *open* annulus). Let $C_{2R} \subset A$ be the circle of radius $2R$. Any two points on C_{2R} are connected by an arc, and any point z in A is connected to a the point $2R\frac{z}{|z|} \in C_{2R}$ by a line segment in A. Thus A is path connected. This implies (since $A \subset U$) that any two points in A are in the same connected component of U. Therefore, any other connected component of U will have no points in A, and thus be bunded by R.

3. I.8, I.10. Also look at I.11 (don't turn it in). *This is a slightly more advanced set of problems* **for learning to work with the differential operators $\bar{\partial}(F) = \frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}$ and Δ (the Laplacian) — we will not focus on this formalism much, but it is useful for people interested in relating complex analysis to other topics in analysis.**

I.8 This problem can be done as an exercise in the multivariable calculus chain rule, once you wrap your head around the potential confusions involved in the unfortunate notation $\partial_z, \partial_{\bar{z}}$. We will use the (better) notation ∂ for $\frac{\partial}{\partial_z}$ and $\bar{\partial}$ for $\frac{\partial}{\bar{z}}$.

First, what are ∂ and $\bar{\partial}$? Like ∂_x, they are *differential operators*, which are operators that accept as input a function f and output another function made out of the derivatives of f. Also like ∂_x, they are *first-order* differential operators, in that they are a linear combination of first-order derivatives of f. Unlike ∂_x, however, ∂ and $\bar{\partial}$ are only defined for complex-valued functions. The importance of these operators in complex analysis is that $\bar{\partial}(f) = 0$ if and only if f is holomorphic (this is Cauchy-Riemann). On the other hand, $\partial(f)$ is just equal to $f'(= \frac{\partial f}{\partial x})$. For *antiholomorphic* functions f (i.e., functions $f(z)$ such that the function $g(z) := f(\bar{z})$ is holomorphic), the roles of ∂ and $\bar{\partial}$ are reversed.

Now for a given vector $\mathbf{v} = (v_0, v_1) \in \mathbb{R}^2$, there is another useful first-order differential operator called the *directional derivative* in the $\mathbf{v}$ direction (this one defined on real functions), namely, $\partial_{\mathbf{v}} := v_0\partial x + v_1\partial_y$. This operator is also useful from the point of view of complex analysis: if $w = a + bi$ is a specific complex number and $\mathbf{w} := (a, b)$ is the corresponding *real* vector, then for a holomorphic function f we have

$$\partial_{\mathbf{w}}(f) \text{ (real vector derivative) } = w \cdot f'.$$

The operator $\bar{\partial}$ is *sort of* like the operator corresponding to the vector $(1, i)$ (and ∂ for $(1, -i)$), but this is not a real vector so it behaves very differently (and this is crucial for the analysis of the complex numbers). Indeed, both $\partial_{\mathbf{w}}(f)$ and $\partial_{\mathbf{w}}(\bar{f})$ will always give nonzero results for any non-constant holomorphic or anti-holomorphic function f.

Moving on to the problem, observe that we have equality of operators

$$\partial_x = \partial + \bar{\partial}$$

(i.e., they give the same answer on any, not necessarily holomorphic, function). Similarly,

$$\partial_y = i(\partial - \bar{\partial}).$$

It follows that if w is a complex number and $\mathbf{w} = (\mathrm{Re}(w), \mathrm{Im}(w))$ is the corresponding *real* two-vector, we have for any function f

$$\partial_{\mathbf{w}} f = w\partial f + \bar{w}\bar{\partial} f.$$

Now the chain rule in multivariable calculus can be elegantly formulated as follows. Suppose $h = f \circ g$ is a composition of two functions $\mathbb{R}^2 \to \mathbb{R}^2$ and $\mathbf{x}_0 \in \mathbb{R}^2$ is some point with $\mathbf{y} = g(\mathbf{x})$. Then

$$\partial_{\mathbf{v}} h(\mathbf{x}) = \partial_{(\partial_v g)} f(\mathbf{y})$$

(here and elsewhere f and its derivatives are evaluated at the value $\mathbf{y} = g(\mathbf{x})$).

In particular plugging in the formula for directional derivative above, we have

$$\partial_x h = \partial_{\partial_x g} f = \partial f \cdot \partial_x g + \bar{\partial} f \cdot \overline{\partial_x g} = \partial f \cdot \partial g + \partial f \cdot \bar{\partial} g + \bar{\partial} f \cdot (\overline{\partial g + \bar{\partial} g}),$$

which turns into

$$\partial_x h = \partial f \partial g + \partial f \bar{\partial} g + \partial f \partial \bar{g} + \partial f \bar{\partial} \bar{g},$$

using the obvious fact that $\bar{\partial}\bar{g} = \overline{\partial g}$ (and vice versa). Similarly, $\partial_y h = \partial f \cdot \partial_y g + \bar{\partial} f \cdot \overline{\partial_y g}$, which expands to

$$\partial_y h = i\left(\partial f \cdot (\partial g - \bar{\partial} g) + \bar{\partial} f \cdot (\bar{\partial}\bar{g} - \partial \bar{g})\right).$$

Taking the combination of the terms above in the formulas $\frac{1}{2}\partial_x h \pm \frac{1}{2i}\partial_y h$ for $\partial h, \bar{\partial} h$ we see (after a bunch of cancellations) the required results. [1]

I.10

$$\partial\bar{\partial} = \frac{1}{2}(\partial_x - i\partial_y) \cdot \frac{1}{2}(\partial_x + i\partial y) = \frac{1}{4}\left(\partial_x^2 + \partial_y^2 + i(\partial_x\partial_y - \partial_y\partial_x)\right).$$

Since $\partial_x\partial_y = \partial_y\partial_x$ (i.e., applying these two operators in any order gives the same answer for any continuously twice differentiable function), this simplifies to $\frac{1}{4}\Delta$, and $\bar{\partial}\partial$ similarly.

4. I.9. Treat this as a standard multivariable calculus problem without worrying about limits, convergence, etc. Note that the Cauchy-Riemann equations in the book split our complex-valued function $F(x, y) := f(x + iy)$ into two functions $u(x, y) + iv(x, y)$.

The first part is a straightforward multivariable calculus problem. Note that the conditions on u, v can be formulated equivalently as a single condition on $f = u + vi$, namely, $\partial_\theta f = ir\partial_r f$.

[1] apologies for the inevitable sign errors.

For the second problem, compute (for $u(z) = \mathrm{Re}\, log(z), v(z) = \mathrm{Im}\, log(z)$) that $\partial_r \log = \frac{1}{r} + 0 \cdot i, \partial_\theta \log = i$, so log satisfies the condition (in the domain indicated).

Notice that $\log(z)$ is a valid antiderivative for $1/z$ in the domain indicated. How do you reconcile this with the fact from class that $1/z$ has no antiderivative because its complex integral around a loop is nonzero? Well, notice that it cannot be extended to a holomorphic — or even to a continuous — function for $\arg(z) = \pi$, as the limits of $\log(z)$ from the two $\arg > \pi$ and $\arg < \pi$ sides differs by a multiple of $2\pi i$.

5. I.16 a-d, I.17 For working with power series.

None of the arguments in these problems will change if you replace a_n by $|a_n|$, so real analysis will be enough here. The answers for 16 are 16a: 1, 16b: 0, 16c: 4, 16d: 27.

6. I.18 Working with power series and changing order of summation.

Say $f(z) = \sum_{n \geq 0} a_n z^n$. Let z_0 be within the disk of convergence, so $|z_0| < R$. Write

$$f(z_0 + z) = \sum_{n \geq 0} a_n (z_0 + z)^n = \sum_{n \geq 0} \sum_{k \leq n} \binom{n}{k} (z^k)(z_0^{n-k}).$$

We want to show that for z in some sufficiently small radius, we can change order of summation. To do this, check absolute convergence. The sum of absolute values is $\sum_{n \geq 0, k \leq n} |a_n| \binom{n}{k} |z|^k |z_0|^{n-k}$ (sums of positive numbers do not depend on order). This simplifies to $\sum_{n \geq 0} |a_n| (|z| + |z_0|)^n$. Since the radius of convergence is the same for the series $\sum a_n z^n$ and $\sum |a_n| z^n$, we see that the double sum of absolute values converges if $|z| < R - |z_0|$ (note that $R - |z_0| > 0$). Thus for z in a disk of positive radius order of summation doesn't matter and

$$\sum a_n (z_0 + z)^n = \sum_{n \geq 0, k \leq n} a_n \binom{n}{k} z_0^{n-k} z^n = \sum_{k \geq 0} (\sum_{n \geq k} a_n \binom{n}{k} z_0^{n-k}) z^k,$$

which is a sum of the form $\sum_{k=0}^{\infty} b_k z^k$, hence a power series. Note that the free coefficient is $\sum a_n z_0^n = f(z_0)$ and the z^1 coefficient is $\sum n a_n z_0^{n-1} = f'(z)$. More generally, you can see by induction (as one would expect from Taylor's theorem) that the z^n coefficient is related to the nth derivative by $b_n = \frac{1}{n!} f^{(n)}(z_0)$.