Math 185 Practice problems for final

May 7, 2020

1 New material

1. (a) Construct a conformal equivalence between the strip $\{x+iy \mid 0 < x+y < \pi$ and the unit disk $\mathbb{D}$.

(b) Construct a conformal equivalence between the "angle" $\{z \in \mathbb{C} \mid z \neq 0, 0 < \arg(z) < \pi/3\}$ and the unit disk $\mathbb{D} \subset \mathbb{C}$.

2. Suppose that $f : S^2 \to S^2$ is a holomorphic (conformal except at finitely many points) function from the Riemann sphere to itself. Let $F : \mathbb{C}- \to \mathbb{C}$ be the (partially defined) corresponding meromorphic function, given (where defined) by $F(z) = P \circ f \circ P_N^{-1}(z)$. Suppose that f is conformal at every preimage point of N. Show that the meromorphic function $F(z)$ is given by the formula $f(z) = az + b + \sum_{k=1}^{n} \frac{a_n}{z - z_k}$, for $a, b, a_1, \ldots, a_n$ complex constants and $z_1, \ldots, z_n$ distinct complex numbers.

3. Define the function $f : \mathbb{C} \to \mathbb{R}^3$ given by

$$f(x+iy) = (\cos x, \sin x, y).$$

Let $Y = \mathrm{Im} f \subset \mathbb{R}^3$. the image of f, be the vertical unit cylinder.

(a) Show that f is a conformal map.

(b) Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ be the set of complex numbers except the origin. Find a (bijective) conformal mapping $g : \mathbb{C}^* \to Y$.

2 Old material

4. Let Ω be a simple closed curve and $p \in \mathrm{Int}\Omega$ a point in its interior. Let $\mathbb{D}$ be the unit disk.

(a) Show that there exists a conformal equivalence (i.e., mapping) from $\Omega \setminus \{p\}$ to $\mathbb{D} \setminus \{0\}$.

(b) Show that there does not exist a conformal equivalence from $\mathbb{C}^*$ to $\mathbb{D} \setminus \{0\}$.

(c) Using problem 3, deduce that there does not exist a conformal equivalence from the cylinder Y to $\mathbb{D} \setminus \{0\}$.

True or false. If true give an argument. If false give a counterexample.

(a) If f and g have a pole at z_0 then $f + g$ has a pole at z_0.

(b) If f and g have a pole at z_0 and both have nonzero residues the fg has a pole at z_0 with a nonzero residue. (c) If f has an essential singularity at $z = 0$ and g has a pole of finite order at $z = 0$ then $f + g$ has an essential singularity at $z = 0$. (d) If $f(z)$ has a pole of order m at $z = 0$ then $f(z^2)$ has a pole of order $2m$.

5. Problem 5. Line integrals (a) Compute $\int_C x dz$ where C is the unit square.
(b) Compute $\int_C \frac{1}{|z|} dz$, where C is the unit circle.
(c) Compute $\int_C \frac{z^2-1}{z^2+1} dz$, where C is the circle of radius 2.
(d) Compute $\int_C \frac{e^z}{z^2} dz$, where C is the circle $|z| = 1$.

6. Suppose f is entire and $|f(z)| > 1$ for all z. Show that f is constant.