Math 185 Topics

March 7, 2020

1 Complex numbers and functions (I.1, Chap. 1 of Gamelin)

Basic question: how to do algebra and calculus with complex numbers?

- Question: How to multiply two complex numbers in polar form,

$$r_1 \exp(i\theta_1) \cdot r_2 \exp(i\theta_2) = r_1 r_2 \exp\left(i(\theta_1 + \theta_2 \mod 2\pi)\right)$$

- Question: How to define $\exp, \sin, \cos$ for complex numbers. Properties of exp.
- Question: How to take the limit $\lim_{z \to z_0} f(z)$ of a complex function.
- What is the complex logarithm $\ln(z)$? Where is it defined? Why is it not the only solution to the equation $\exp(a + bi) = z$ and what are all the solutions in terms of polar coordinates (r, θ) for z?

2 Complex derivatives and holomorphicity basics (I.2)

Basic question: what are holomorphic functions? What are some examples?

- Question: What is a complex derivative? When does it exist?
- **Cauchy-Riemann Theorem**: holomorphicity implies existence of (continuous) partial derivatives. Conversely, existence of (continuous) partial derivatives does not imply holomorphicity. We need to impose the Cauchy-Riemann relation, $\partial_y f = i\partial_x f$. Think: "rate of change in the i direction is i times the rate of change in the 1 direction".
- **Analytic functions are holomorphic.** Intuition of proof: the complex derivative of any partial sum $\sum_{k=0}^{N} a_k (z - z_0)^k$ is equal to $\sum_{k=1}^{N} k a_k z^{k-1}$. In particular, $\sum_{k=0}^{N} a_k (z - z_0)^k$ are holomorphic functions and their derivatives converge to $\sum_{k=1}^{\infty} j a_k z^{k-1}$ within the radius of convergence. To prove

the the theorem, we need to be a little more careful with convergence (specifically: control the error terms that show up in the computation of the derivative), but you don't need to remember how to do this.

3 Path integrals and antiderivatives (I.3, II.1, II.2)

Questions.

- What is the definition of $\int_\gamma f(z)dz$ for γ a path from $a \in \mathbb{C}$ to $b \in \mathbb{C}$?
- What is $\int_\gamma f(z)dz$ when f has a holomorphic antiderivative F? (Answer: it is $F(b)-F(a)$, this is the complex chain rule applied to the composition $F \circ \gamma$.)
- Does this hold if we don't assume f has an antiderivative in Ω? (Answer: not necessarily.)

Statements.

- The functions z^n for any integer $n \neq -1$ have antiderivatives, so they are easy to integrate along a path. In particular, their integral along a loop equal to zero. (Why?)
- The fact $\oint_{C_1} \frac{1}{z} = 2\pi i$ implies that $\frac{1}{z}$ cannot have an antiderivative on any domain that contains the circle C_1. (Why?)
- Key theorem: **Cauchy's Theorem**: if γ is a simple closed loop and f is holomorphic on (a domain containing) γ and its interior, then $\oint_\gamma f(z)dz = 0$. Derived from the (extremely similar) **Goursat's Theorem**, which is Cauchy's theorem for a simple polygon (rectangle in class; triangle in book. You can use any version.)
- Corollaries of Cauchy:
  1. **Antiderivative theorem: a function which is holomorphic on the interior of a simple closed loop has a holomorphic antiderivative in the interior of this loop.**
  2. If f is holomorphic on a simple closed loop γ but $\oint_\gamma f(z)dz \neq 0$ then f must not be holomorphic (e.g. have a pole) somewhere in the interior of γ.

4 Cauchy integral formula and Analyticity (II.4, Morera from II.5)

Questions.

- How can we turn the seeming bug in complex analysis, that the change of variables formula fails for loops, into a feature?
- How can we express a value or a derivative of $f(z_0)$ in terms of values of f "far away" from z_0?
- When does the converse of "holomorphic $\implies$ analytic" hold? (Answer: always, because of Cauchy integral magic.)

Statements.

- **Cauchy Integral formula** If f is holomorphic on (a domain containing) γ and the interior of γ then

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0}dz$$

  for z_0 is in the interior of γ (but not on γ itself or else the path integral is undefined!).

- **Generalization** More generally, we can compute any derivative $f^{(n)}(z_0)$ in terms of a path integral using the following formula:

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}dz.$$

- **Liouville:** A consequence of Cauchy for first derivatives: a function which is bounded and *entire* (everywhere holomorphic) must be constant. **Corollary:** fundamental theorem of arithmetic (every polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$).

- **Holomorphic functions are analytic.** Notice that $\frac{f^{(n)}(z_0)}{n!}$ is the nth Taylor coefficient of f around z_0. A real infinitely differentiable function has a Taylor series, but might not be equal to the analytic function defined by this series. A holomorphic function, however, satisfies $f(z) = \sum a_n(z-z_0)^n$ for $a_n = \frac{f^{(n)}(z_0)}{n!}$ within a nonzero radius of convergence. We show this by expanding the term $\frac{1}{(z-z_0)^{n+1}}$ in the Cauchy theorem as a geometric series. This is one of the important ways in which complex analysis is "magic".

- **Morera** A consequence of analyticity is *Morera's theorem*: a continuous function (on a domain Ω) whose integral over any simple closed loop is 0 must be holomorphic. Note that the converse is not necessarily true, since Ω might not be simply connected (it might have a "hole"). But if Ω is, for example, the interior of a simple closed curve, then the converse is true by Cauchy's theorem.

5 Poles and residues (III.1, III.2)

If f is defined on a domain Ω that contains all the points in some disk around z_0 except for z_0 itself (sometimes called a *punctured neighborhood of* z_0), then we say f has an (at worst) isolated singularity at z_0. The singularity **removable** if f can be extended to z_0 holomorphically (in which case we say f is not singular at z_0). It is **a pole** if f is singular at z_0 but f^{-1} (here understood as $\frac{1}{f(z)}$) is not (in which case f^{-1} must have a zero at z_0). An isolated singularity which is neither removable nor a pole is called an **essential singularity**.

Statements.

- A function f which is holomorphic at z_0 has a *zero of order* n if $f(z) = (z - z_0)^n \tilde{f}(z)$ for $\tilde{f}$ a function which is holomorphic and nonzero (a.k.a. invertible) at z_0. (If $f(z_0) \neq 0$ we say f has a "zero of order zero" at z_0.)

- A function f with a singularity at z_0 has a *pole of order* n if f^{-1} has a zero of order n.

- If f has a zero of order n it has a Taylor series $f(z) = a_n(z - z_0)^n + O(z - z_0)^{n+1}$. If f has a pole of order n then $f(z)$ has a Laurent series, $f(z) = a_{-n}(z - z_0)^{-n} + O(z - z_0)^{-(n-1)}$. The finite sum $\sum_{k=-n}^{-1} a_k z^k$ is called the singular part, also known as the principal part. And the (holomorphic at z_0) infinite sum $\sum_{k=0}^{\infty} a_k z^k$ is called the holomorphic part.

- The most important term in the principal part is the *residue*, $\mathrm{Res}_{z_0}(f) = a_{-1}$ (for $f = \sum a_k(z - z_0)^k$ the Laurent expansion).

- A key formula: if f has a *simple pole*, i.e. a pole of order 1, at z_0 then $f(z) = a_{-1}(z-z_0)^{-1}+O(1)$ and $f^{-1} = a_1^{-1}(z-z_0)^1+O(z-z_0)^2$. Therefore if we write $g(z) = f^{-1}(z)$ and it has a power series expansion $g(z) = \sum b_k(z - z_0)^k$, then $a_{-1} = b_1^{-1}$ (in particular, it is never 0). Alternatively: If f has a simple pole at z_0, then

$$\mathrm{Res}_{z_0} f = \left(\frac{1}{f}\right)'(z_0).$$

- **The residue formula** Assume f is defined on (a domain Ω that contains) γ and also on the interior of γ except for at finitely many points $z_1, \ldots, z_n$, all of which are poles of f. Then $\oint_\gamma f(z)dz = 2\pi i \cdot \left(\sum_{k=1}^{n} \mathrm{Res}_{z_k} f(z)\right)$. This formula is obvious from the Laurent series expression if there is one pole z_1, and if there are multiple poles can be obtained either as a keyhole contour argument or by observing that the sum of the singular parts of f at the z_k exactly cancels the singularities of f.

6 Toy contours, keyholes and proving an $R \to \infty$ integral goes to 0 (II.3 and other places).

- It is sometimes useful to "split" a contour into smaller contours, using cancellation of the path integral over a segment and the same segment going in the opposite direction. We used this, for example, in our proof of Goursat's theorem. Sometimes it is useful instead to take a pair of segments ϵ apart which almost cancel, and to observe that they cancel in the limit (this is the keyhole argument).

- More generally, many integrals can be reduced to a path integral by taking a limit of contours depending on a parameter R, taking this parameter to ∞, and noticing that certain contour integrals go to zero. Useful facts:

  1. For $n \geq 2$, the contour integral of a function of the form $\frac{1}{z^n}$ (and more generally, the inverse to a polynomial of degree n) will go to 0 over any arc of a circle of radius $R \to \infty$.
  2. As $y \to \infty$, the function e^{-x+iy_0} goes to 0 exponentially while $|e^{x+iy_0}|$ goes to ∞ exponentially in the $x \to \infty$ limit.
  3. Both $|\sin(e^{x_0+iy})|$ and $|\cos(e^{x_0+iy})|$ go to ∞ exponentially in the $y \to \pm\infty$ limit (since they have both a e^{-y+ix_0} and an e^{y-ix_0} term).