Math 185 Final Topics

May 4, 2020

1 New material

1.1 Conformal functions and mappings: complex analysis in a more general geometric context

Useful source: `https://math.mit.edu/~jorloff/18.04/notes/topic10.pdf`.

- A conformal function is a function which preserves angles. A conformal mapping is a bijective conformal function.
- Two surfaces X, Y are said to be conformally equivalent if a conformal *mapping* $f : X \to Y$ exists.
- A conformal mapping $f : \Omega \to \Omega'$ has an inverse, which is also conformal.
- Statements: For two complex domains Ω and Ω', a function $f : \Omega \to \Omega'$ is conformal if and only if f is holomorphic and for all $z \in \Omega$, we have $f'(z) \neq 0$.
- A conformal mapping $f : \Omega \to S^2$ (for S^2 the Riemann sphere) is equivalent to the data of a meromorphic function (function with isolated singularities which are poles) $F : \Omega- \to \mathbb{C}$ such that F has at worst simple poles and $F'(z) \neq 0$ when $z \in \Omega$ is not a pole. The correspondence is via $f(z) := P_N^{-1}(F(z))$, with $f(z) := N$ (the north pole, corresponding to the point at "infinity") for z a pole.
- A conformal function $f : S^2 \to X$ (for X any domain in the plane, the sphere, or any other surface) is a continuous function $f : S^2 \to X$ for which both $f \circ P_N$ and $f \circ \bar{P}_S$ are conformal functions from $\mathbb{C}$ to X.
- Any conformal function $f : S^2 \to S^2$ is the extension of a fractional linear transformation.
- For $\mathbb{D} \subset \mathbb{C}$ the unit disk Any conformal function $f : \mathbb{D} \to \mathbb{D}$ is also a fractional linear transformation, of a particular kind (specifically: $z \mapsto \frac{z-\alpha}{1-\bar{\alpha}\cdot z}$ for some complex number α). There exists a conformal function (indeed, many) that take any point of the disk to any other point.

- **Riemann mapping theorem.** The interior of any simple closed curve in $\mathbb{C}$ (and more generally, any proper simply-connected open domain in $\mathbb{C}$) is conformally equivalent to the disk.
- **Uniformization theorem** (for the disk): any surface $X \subset \mathbb{R}^3$ (or more generally in $\mathbb{R}^n$) which is smoothly topologically equivalent to the open disk (i.e., parametrized by a disk with injective Jacobian at every point) is conformally equivalent either to the disk $\mathbb{D}$ or the plane $\mathbb{C}$.

1.2 The argument principle

- A good source with more details: `https://math.mit.edu/~jorloff/18.04/notes/topic11.pdf`.
- For a function $f : \Omega \to \mathbb{C}$, its *logarithmic derivative* is defined as $\frac{\text{dlog} f}{dz} := \frac{f'}{f}$. It satisfies $\text{dlog}(fg)dz = \frac{\text{dlog} f}{dz} + \frac{\text{dlog} g}{dz}$.
- Main result: if $f : \Omega- \to \mathbb{C}$ is a meromorphic function (function with isolated singularities which are poles) and γ is a simple closed curve in Ω, then $\oint_\gamma \frac{\text{dlog} f}{dz} = 2\pi i(Z - P)$, where Z is the number of zeroes and P the number of poles in the interior of γ, *counted with multiplicity.*

2 Old material

2.1 Complex numbers and functions (I.1, Chap. 1 of Gamelin)

Basic question: how to do algebra and calculus with complex numbers?

- Question: How to multiply two complex numbers in polar form,

$$r_1 \exp(i\theta_1) \cdot r_2 \exp(i\theta_2) = r_1 r_2 \exp\left(i(\theta_1 + \theta_2 \mod 2\pi)\right)$$

- Question: How to define $\exp, \sin, \cos$ for complex numbers. Properties of exp.
- Question: How to take the limit $\lim_{z\to z_0} f(z)$ of a complex function.
- What is the complex logarithm $\ln(z)$? Where is it defined? Why is it not the only solution to the equation $\exp(a + bi) = z$ and what are all the solutions in terms of polar coordinates (r, θ) for z?

2.2 Complex derivatives and holomorphicity basics (I.2)

Basic question: what are holomorphic functions? What are some examples?

- Question: What is a complex derivative? When does it exist?

- **Cauchy-Riemann Theorem**: holomorphicity implies existence of (continuous) partial derivatives. Conversely, existence of (continuous) partial derivatives does not imply holomorphicity. We need to impose the Cauchy-Riemann relation, $\partial_y f = i\partial_x f$. Think: "rate of change in the i direction is i times the rate of change in the 1 direction".

- **Analytic functions are holomorphic.** Intuition of proof: the complex derivative of any partial sum $\sum_{k=0}^{N} a_k(z-z_0)^k$ is equal to $\sum_{k=1}^{N} ka_k z^{k-1}$. In particular, $\sum_{k=0}^{N} a_k(z-z_0)^k$ are holomorphic functions and their derivatives converge to $\sum_{k=1}^{\infty} ja_k z^{k-1}$ within the radius of convergence. To prove the the theorem, we need to be a little more careful with convergence (specifically: control the error terms that show up in the computation of the derivative), but you don't need to remember how to do this.

2.3 Path integrals and antiderivatives (I.3, II.1, II.2)

Questions.

- What is the definition of $\int_\gamma f(z)dz$ for γ a path from $a \in \mathbb{C}$ to $b \in \mathbb{C}$?

- What is $\int_\gamma f(z)dz$ when f has a holomorphic antiderivative F? (Answer: it is $F(b)-F(a)$, this is the complex chain rule applied to the composition $F \circ \gamma$.)

- Does this hold if we don't assume f has an antiderivative in Ω? (Answer: not necessarily.)

Statements.

- The functions z^n for any integer $n \neq -1$ have antiderivatives, so they are easy to integrate along a path. In particular, their integral along a loop equal to zero. (Why?)

- The fact $\oint_{C_1} \frac{1}{z} = 2\pi i$ implies that $\frac{1}{z}$ cannot have an antiderivative on any domain that contains the circle C_1. (Why?)

- Key theorem: **Cauchy's Theorem**: if γ is a simple closed loop and f is holomorphic on (a domain containing) γ and its interior, then $\oint_\gamma f(z)dz = 0$. Derived from the (extremely similar) **Goursat's Theorem**, which is Cauchy's theorem for a simple polygon (rectangle in class; triangle in book. You can use any version.)

- Corollaries of Cauchy:

  1. **Antiderivative theorem: a function which is holomorphic on the interior of a simple closed loop has a holomorphic antiderivative in the interior of this loop.**
  2. If f is holomorphic on a simple closed loop γ but $\oint_\gamma f(z)dz \neq 0$ then f must not be holomorphic (e.g. have a pole) somewhere in the interior of γ.

2.4 Cauchy integral formula and Analyticity (II.4, Morera from II.5)

Questions.

- How can we turn the seeming bug in complex analysis, that the change of variables formula fails for loops, into a feature?
- How can we express a value or a derivative of $f(z_0)$ in terms of values of f "far away" from z_0?
- When does the converse of "holomorphic $\implies$ analytic" hold? (Answer: always, because of Cauchy integral magic.)

Statements.

- **Cauchy Integral formula** If f is holomorphic on (a domain containing) γ and the interior of γ then

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} dz$$

  for z_0 is in the interior of γ (but not on γ itself or else the path integral is undefined!).

- **Generalization** More generally, we can compute any derivative $f^{(n)}(z_0)$ in terms of a path integral using the following formula:

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

- **Liouville:** A consequence of Cauchy for first derivatives: a function which is bounded and *entire* (everywhere holomorphic) must be constant. **Corollary:** fundamental theorem of arithmetic (every polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$).

- **Holomorphic functions are analytic.** Notice that $\frac{f^{(n)}(z_0)}{n!}$ is the nth Taylor coefficient of f around z_0. A real infinitely differentiable function has a Taylor series, but might not be equal to the analytic function defined by this series. A holomorphic function, however, satisfies $f(z) = \sum a_n(z - z_0)^n$ for $a_n = \frac{f^{(n)}(z_0)}{n!}$ within a nonzero radius of convergence. We show this by expanding the term $\frac{1}{(z-z_0)^{n+1}}$ in the Cauchy theorem as a geometric series. This is one of the important ways in which complex analysis is "magic".

- **Morera** A consequence of analyticity is *Morera's theorem*: a continuous function (on a domain Ω) whose integral over any simple closed loop is 0 must be holomorphic. Note that the converse is not necessarily true, since Ω might not be simply connected (it might have a "hole"). But if Ω is, for example, the interior of a simple closed curve, then the converse is true by Cauchy's theorem.

2.5 Poles and residues (III.1, III.2)

If f is defined on a domain Ω that contains all the points in some disk around z_0 except for z_0 itself (sometimes called a *punctured neighborhood of* z_0), then we say f has an (at worst) isolated singularity at z_0. The singularity **removable** if f can be extended to z_0 holomorphically (in which case we say f is not singular at z_0). It is **a pole** if f is singular at z_0 but f^{-1} (here understood as $\frac{1}{f(z)}$) is not (in which case f^{-1} must have a zero at z_0). An isolated singularity which is neither removable nor a pole is called an **essential singularity**.

Statements.

- A function f which is holomorphic at z_0 has a *zero of order* n if $f(z) = (z - z_0)^n \tilde{f}(z)$ for $\tilde{f}$ a function which is holomorphic and nonzero (a.k.a. invertible) at z_0. (If $f(z_0) \neq 0$ we say f has a "zero of order zero" at z_0.)

- A function f with a singularity at z_0 has a *pole of order* n if f^{-1} has a zero of order n.

- If f has a zero of order n it has a Taylor series $f(z) = a_n(z - z_0)^n + O(z - z_0)^{n+1}$. If f has a pole of order n then $f(z)$ has a Laurent series, $f(z) = a_{-n}(z - z_0)^{-n} + O(z - z_0)^{-(n-1)}$. The finite sum $\sum_{k=-n}^{-1} a_k z^k$ is called the singular part, also known as the principal part. And the (holomorphic at z_0) infinite sum $\sum_{k=0}^{\infty} a_k z^k$ is called the holomorphic part.

- The most important term in the principal part is the *residue*, $\mathrm{Res}_{z_0}(f) = a_{-1}$ (for $f = \sum a_k (z - z_0)^k$ the Laurent expansion).

- A key formula: if f has a *simple pole*, i.e. a pole of order 1, at z_0 then $f(z) = a_{-1}(z-z_0)^{-1} + O(1)$ and $f^{-1} = a_1^{-1}(z-z_0)^1 + O(z-z_0)^2$. Therefore if we write $g(z) = f^{-1}(z)$ and it has a power series expansion $g(z) = \sum b_k (z - z_0)^k$, then $a_{-1} = b_1^{-1}$ (in particular, it is never 0). Alternatively: If f has a simple pole at z_0, then

$$\mathrm{Res}_{z_0} f = \left(\frac{1}{f}\right)' (z_0).$$

- **The residue formula** Assume f is defined on (a domain Ω that contains) γ and also on the interior of γ except for at finitely many points $z_1, \ldots, z_n$, all of which are poles of f. Then $\oint_\gamma f(z) dz = 2\pi i \cdot \left(\sum_{k=1}^{n} \mathrm{Res}_{z_k} f(z)\right)$. This formula is obvious from the Laurent series expression if there is one pole z_1, and if there are multiple poles can be obtained either as a keyhole contour argument or by observing that the sum of the singular parts of f at the z_k exactly cancels the singularities of f.

2.6 Toy contours, keyholes and proving an $R \to \infty$ integral goes to 0 (II.3 and other places).

- It is sometimes useful to "split" a contour into smaller contours, using cancellation of the path integral over a segment and the same segment going in the opposite direction. We used this, for example, in our proof of Goursat's theorem. Sometimes it is useful instead to take a pair of segments ϵ apart which almost cancel, and to observe that they cancel in the limit (this is the keyhole argument).

- More generally, many integrals can be reduced to a path integral by taking a limit of contours depending on a parameter R, taking this parameter to ∞, and noticing that certain contour integrals go to zero. Useful facts:

  1. For $n \geq 2$, the contour integral of a function of the form $\frac{1}{z^n}$ (and more generally, the inverse to a polynomial of degree n) will go to 0 over any arc of a circle of radius $R \to \infty$.
  2. As $y \to \infty$, the function e^{-x+iy_0} goes to 0 exponentially while $|e^{x+iy_0}|$ goes to ∞ exponentially in the $x \to \infty$ limit.
  3. Both $|\sin(e^{x_0+iy})|$ and $|\cos(e^{x_0+iy})|$ go to ∞ exponentially in the $y \to \pm\infty$ limit (since they have both a e^{-y+ix_0} and an e^{y-ix_0} term).