Math 185 Practice problems for final

May 9, 2020

1 New material

1. (a) Construct a conformal equivalence between the strip $\{x + iy \mid 0 < x + y < \pi$ and the unit disk $\mathbb{D}$.

First construct an equivalence bewtween the the strip and the horizontal strip $0 < y < \pi :$ we can do this by $f_1(z) = (1+i)z$. Then apply $f_2 : z \mapsto \exp(z)$, to get from the strip to the half-plane. Finally, apply $f_3 : z \mapsto \frac{z-i}{z+i}$, a to identify the upper half-plane with the disk $\mathbb{D}$. So the composed map is $f_3 \circ f_2 \circ f_1$, which is

$$z \mapsto f_3\left(\exp\left((1+i)z\right)\right),$$

for f_3 the fractional linear transformation above.

(b) Construct a conformal equivalence f between the "angle" $\{z \in \mathbb{C} \mid z \neq 0, 0 < \arg(z) < \pi/3\}$ and the unit disk $\mathbb{D} \subset \mathbb{C}$.

Start with the map $z \mapsto z^3$ for a bijective map between Ω and the upper half-plane $\mathbb{H}$. Then apply the conformal equivalence f_3 from the half-plane to the disk. We get the composition $f : z \mapsto \frac{z^3-i}{z^3+i}$..

2. Suppose that $f : S^2 \to S^2$ is a holomorphic (conformal except at finitely many points) function from the Riemann sphere to itself. Let $F : \mathbb{C}- \to \mathbb{C}$ be the (partially defined) corresponding meromorphic function, given (where defined) by $F(z) = P \circ f \circ P_N^{-1}(z)$. Suppose that f is conformal at every preimage point of N. Show that the meromorphic function $F(z)$ is given by the formula $f(z) = az + b + \sum_{k=1}^{n} \frac{a_n}{z-z_k}$, for $a, b, a_1, \ldots, a_n$ complex constants and $z_1, \ldots, z_n$ distinct complex numbers.

Let F be the (partially defined) function $F = P_N^{-1} \circ f \circ P_N$. Let $p \in S^2 \setminus N$, and $z_0 = P_N(p)$. Then $f(p) = N$ if and only if F has a pole at z_0. For such z_0, we have f is conformal at z_0 if and only if $\frac{1}{F}'(z) \neq 0$, i.e. if F has a simple pole at z_0. We also know that $f(N) = N$ (i.e., $\lim_{|z|\to\infty} |F(z)| = \infty$.) Conformality at N (corresponding to ∞) is equivalent to the requirement that $F(1/z)$ has a simple pole at 0, i.e. for $G(w) = F(1/w)$, we are requiring for $\lim_{z\to 0} zG(z)$ to exist (and be nonzero).

Now let $z_1, z_2, \dots, z_n$ be the residues of F at all preimages of N other than N itself (there are finitely many since a closed subset of S^2 is compact). Let a_k be the residue at z_k (equivalently, the leading term of the Laurent series). Define $\tilde{F}(z) := F(z) - \sum \frac{a_k}{z-z_k}$. Then $\tilde{F}(z)$ is a now holomorphic function, and since $\lim_{z\to\infty} \frac{a_k}{z-z_k} = 0$, we deduce that $\lim_{|z|\to\infty} \frac{\tilde{F}(z)}{z} = \lim_{|z|\to\infty} \frac{F}{z}$, which is a finite number. Thus $\tilde{F}(z)$ is a holomorphic function of linear growth. We can now apply the Cauchy inequalities to deduce that the second derivative $\tilde{F}^{(2)}(z) = 0$ for any z, i.e. $\tilde{F}$ is linear.[1]

So $\tilde{F}(z) = az + b$ and (by definition of $\tilde{F}$) we have

$$F(z) = \tilde{F}(z) + \sum_{k=1}^{n} \frac{a_k}{z - z_k},$$

as desired.

3. Define the function $f : \mathbb{C} \to \mathbb{R}^3$ given by

$$f(x + iy) = (\cos x, \sin x, y).$$

Let $Y = \mathrm{Im} f \subset \mathbb{R}^3$. the image of f, be the vertical unit cylinder.

(a) Show that f is a conformal map. Let $\vec{v} := \frac{df}{dx}$ and $\vec{w} := \frac{df}{dy}$. Then $\vec{v} = (-\sin(x), \cos(x), 0)$ and $\vec{y} = (0, 0, 1)$. These are orthogonal vectors of the same length, so the derivative of f takes an orthonormal basis to the rescaling of an orthonormal pair of vectors. Hence it preserves angles.

(b) Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ be the set of complex numbers except the origin. Find a (bijective) conformal mapping $g : \mathbb{C}^* \to Y$. Compare with the conformal map $\exp(iz) : \mathbb{C} \to \mathbb{C}^*$. Since both $f : \mathbb{C} \to Y$ and $\exp(iz) : \mathbb{C} \to \mathbb{C}^*$ are conformal and periodic with period 2π, we can check whether the composition $f(\log(z))$ extends to the entire $\mathbb{C}^*$. Indeed, we obtain the function $g : (r, \theta) \mapsto (\sin\theta, \cos\theta, \log(r))$, which is well-defined and continuous for all $r > 0, \theta \in [0, 2\pi)$ (continuous since neither sin nor cos change when θ changes by 2π).

In fact conformality for g can be seen to follow from conformality of (an appropriately chosen branch of) log and conformality of f. We check it directly: we use that the curves $\gamma_r(t) := (\theta, r+t)$ and $\gamma_\theta(t) := (r, \theta + t/r)$ have derivatives $v = \hat{\theta}$ and $w = \hat{\theta} \cdot i$, respectively, which form an orthogonal basis. Now $\frac{df}{d\gamma_\theta} = \frac{(\cos\theta, \sin\theta, 0)}{r}$ and $\frac{df}{d\gamma_t} = \frac{(0,0,1)}{r}$, so a pair of orthogonal vectors gets sent to a pair of orthogonal vectors of equal length $(1/r)$. This implies that angles are conserved.

4. Let Ω be a simple closed curve and $p \in \mathrm{Int}\Omega$ a point in its interior. Let $\mathbb{D}$ be the unit disk.

[1] Alternatively to see $\tilde{F}$ is linear: we can observe that $\frac{\tilde{F}}{z} - \frac{\tilde{F}(0)}{z}$ is an entire, bounded function, hence constant.

(a) Show that there exists a conformal equivalence (i.e., mapping) from $\Omega \setminus \{p\}$ to $\mathbb{D} \setminus \{0\}$.

The Riemann mapping theorem says that there is a conformal mapping $f : \Omega \to \mathbb{D}$. It might not take p to 0. Suppose $f(p) = w$. Let $f_w : \mathbb{D} \to \mathbb{D}$ be the conformal map $f_w : z \mapsto \frac{z-w}{1-\bar{w}z}$. Then composing with f_w sends w is a conformal mapping from $\mathbb{D}$ to itself which takes w to 0. Set $g = f_w \circ f$. Then g maps $\Omega \setminus p$ bijectively to $\mathbb{D} \setminus 0$, giving the desired map.

(b) Show that there does not exist a conformal equivalence from $\mathbb{C}^*$ to $\mathbb{D} \setminus \{0\}$.

Assume $f : \mathbb{C}^* \to \mathbb{D} \setminus 0$ is a conformal mapping. Define $h(z) = f(\exp(z))$. Then h is an entire bounded function. By Liouville, h is constant. This contradicts f being a bijection.

(c) Using problem 3, deduce that there does not exist a conformal equivalence from the cylinder Y to $\mathbb{D} \setminus \{0\}$.

Assume $\alpha : Y \to \mathbb{D} \setminus 0$ is such a mapping. Let $g : \mathbb{C}^* \to Y$ be the conformal mapping from problem 3b. Then $\alpha \circ g$ would contradict (c). (Alternatively: let $f : \mathbb{C} \to Y$ be the map from 3a. Then $\alpha \circ f$ would be an entire bounded function, contradicting α being conformal.

2 Old material

5. (a) If f and g have a pole at z_0 then $f + g$ has a pole at z_0. False: take $f = 1/z$ and $g = -1/z$.

(b) **If f and g have a pole at z_0 and both have nonzero residues the fg has a pole at z_0 with a nonzero residue.** False. Take $f = 1/z$ and $g = 1/z$. (However taking e.g. $f = 1 + 1/z$ and $g = 1/z$ would create a nonzero residue!)

(c) If f has an essential singularity at $z = 0$ and g has a pole of finite order at $z = 0$ then $f + g$ has an essential singularity at $z = 0$.

True. (Equivalent to the sum of two functions with at worst a pole at $z = 0$ having sum with at worst a pole.)

(d) If $f(z)$ has a pole of order m at $z = 0$ then $f(z^2)$ has a pole of order $2m$.

True. Look at leading Laurent coefficient.

6. Line integrals (a) Compute $\int_C x dz$ where C is the unit square bounding $\{(x + iy \mid 0 \leq x, y \leq 1\}$. Answer: $-i$.

(b) Compute $\int_C \frac{1}{|z|} dz$, where C is the unit circle. Answer: 0.

(c) Compute $\int_C \frac{z^2-1}{z^2+1} dz$, where C is the circle of radius 2. Answer: 0 (the two residues cancel.)

(d) Compute $\int_C \frac{e^z}{z^2} dz$, where C is the circle $|z| = 1$. Answer: $2\pi i$

6. Suppose f is entire and $|f(z)| > 1$ for all z. Show that f is constant. Apply Liouville to $1/f$.