List of "facts" about conformal maps

May 12, 2020

1. A conformal function is one that preserves angles.

2. A conformal mapping is a conformal function that is a bijection.

3. To check the conformal property for a function $f : X \to Y$ at a point $p \in X$, find an orthonormal basis $\vec{v}_1, \vec{v}_2$ in the tangent space of X. Then f is conformal at p if and only if $J_p f(\vec{v}_1), J_p f(\vec{v}_2)$ are orthogonal vectors of the same length. Here $J_p f$ is the Jacobian of f. Equivalently, if and only if for some pair of curves γ_1, γ_2 with orthonormal tangent vectors $v_1 := \frac{d\gamma_1}{dt}(0), v_2 := \frac{d\gamma_2}{dt}(0)$, the vectors $\frac{df \circ \gamma_1}{dt}(0)$ and $\frac{df \circ \gamma_2}{dt}(0)$ are orthogonal and have the same nonzero length.[1]

4. In the special case $X \subset \mathbb{C}$, the above criterion for $f : X \to Y$ to be conformal is equivalent to $\frac{df}{dx}, \frac{df}{dy}$ being orthogonal, nonzero, and having the same length.

5. The composition of two conformal functions (respectively, mappings) is a conformal function (respectively, mapping). A confomral mapping always has an inverse, which is also confromal.

6. For $\Omega \subset \mathbb{C}$ an open domain, a function $f : \Omega \to \mathbb{C}$ (or $f : \Omega \to \Omega'$ for Ω' also in $\mathbb{C}$) is conformal at z_0 if and only if it is holomorphic and $f'(z_0) \neq 0$. In particular, it is conformal if the property $f'(z_0) \neq 0$ holds for all $z_0 \in \Omega$.

7. A conformal function $f : \Omega \to S^2$ is the same data as a meromorphic function $F : \Omega - \to \mathbb{C}$ (i.e., a function which is either holomorphic or has isolated singularities all of which are poles), whose poles are simple and which has nonzero derivative at all points $z \in \Omega$ which are not a pole. The relationship is $F(z) = P_N(f(z))$, with poles precisely when $f(z) = N$.

8. For any surface X, a conformal map $f : S^2 \to X$ is equivalent to a map $F : \mathbb{C} \to X$ such that F is conformal and the function $G(z) := F(1/z)$ (a priori defined on $\mathbb{C} \setminus \{0\}$) extends to a conformal function on the entire plane $\mathbb{C}$. Here f is related to F and G by $F = f \circ P_N^{-1}$ and $G = f \circ \bar{P}_S^{-1}$

[1] Here and in the subsequent parts I am being sloppy about the difference between conformal and *anticonformal*, i.e. angle-preserving and angle-reversing. You can safely ignore this difference for the purposes of the exam.

(here P_N and $\bar{P}_S$ are Northern and Southern stereographic projection, respectively).

9. In particular, a conformal function $S^2 \to S^2$ is equivalent to a meromorphic function F with simple poles and nonzero derivatives such that the function $G = F(1/z)$ is also meromorphic and has either a simple pole or a nonzero (note that if F has a simple pole or a finite value with nonzero derivative at z_0 and $z_0 \neq 0$ then G has a simple pole or finite value with nonzero derivative at $1/z_0$. This means that the only value where the properties of G need to be verified is near $z = 0$.) We have seen that any such map is in fact associated to a fractional linear transformation.

10. A holomorphic function $\Omega \to S^2$ (defined as a continuous function which is conformal except possibly at finitely many points) is equivalent to a meromorphic function on Ω (i.e. a function with at worst isolated singularities all of which are poles).

11. A holomorphic function $S^2 \to S^2$ is then determined by a meromorphic function $F : \mathbb{C} - \to \mathbb{C}$ such that $F(1/z)$ is also meromorphic.[2]

12. Any simply connected open $\Omega \subset \mathbb{C}$ which is not $\mathbb{C}$ itself is conformally equivalent to the unit disk $\mathbb{D}$. (We showed this more particularly for Ω which are cut out by simple closed curves.)

13. Any map from $\mathbb{D}$ to itself is called a conformal automorphism, and must have the form $e^{i\theta} f_\alpha$, for f_α the fractional linear transformation

$$f_\alpha : z \mapsto \frac{z - \alpha}{1 - \bar{\alpha} z}.$$

(This is sometimes known as the Schwartz lemma.)

14. The fractional linear transformation $g : z \mapsto \frac{z-i}{z+i}$ gives a conformal mapping from the unit disk $\mathbb{D}$ to the conformal plane $\mathbb{H}$ (defined as $\mathbb{H} = \{z \mid \mathrm{Im}(z) > 0\}$). Any other such tranformation $\tilde{g} : \mathbb{D} \to \mathbb{H}$ will be related to g by $\tilde{g} = g \circ f$ for $f : \mathbb{D} \to \mathbb{D}$ a conformal automorphism (to see this: take $f = g^{-1} \circ \tilde{g}$).

15. A conformal automorphism $h : \mathbb{H} \to \mathbb{H}$ is given by the fractional linear transformation $z \mapsto \frac{az+b}{cz+d}$, for a, b, c, d real numbers satisfying $\begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0$.

16. Other useful conformal mappings:

    (a) $\exp(z)$ between the strip $0 < \mathrm{Im}(z) < \pi$ and $\mathbb{H}$,

    (b) $\exp(z)$ between the strip $-\pi < \mathrm{Im}(z) < \pi$ and $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$

    (c) $\log(z)$, the inverse of the above mapping.

[2]Though we did not show this, it can be shown that any such function is *rational*, i.e. the quotient of two nonzero polynomials.