Math 185 Final Exam May 12, 2020

Question	Points	Score
1	20	
2	25	
3	20	
4	20	
5	20	
6	20	
Total:	125	

1. (a) (5 points) Let f be a holomorphic function defined in the upper half-plane $\mathbb{H} = \{z \in \mathbb{C}, \mathrm{Im}(z) > 0\}$, which is periodic with period 2π (i.e., $f(z+2\pi) = f(z)$) *and such that $f(z)$ is bounded on $\mathbb{H}$*. Show that f can be expressed as

$$f(z) = \sum_{n=0}^{\infty} a_n \exp(n \cdot i \cdot z),$$

for $a_n \in \mathbb{C}$, with $|a_n|^{1/n}$ bounded.[1]

   (b) (5 points) Show that moreover, we have $\limsup |a_n|^{1/n} < \frac{1}{r}$ for $r > 1$ a real number if and only if f can be holomorphically extended to a holomorphic function g defined on the plane $\{z \mid \mathrm{Im}(z) > -\log(r)\}$.[2]

   (c) (10 points) Suppose that f is a (bounded for $\mathrm{Im}(z)$ sufficiently large) holomorphic function defined for $\mathrm{Im}(z) > -\epsilon$, except for isolated singularities which are poles at the set of points of the form $z_0 + 2n\pi$ for $n \in \mathbb{Z}$ (here z_0 is a fixed complex number with $\mathrm{Im}(z_0) > 0$). Show that f can be expressed as

$$f(z) = P\left(\mathrm{cotan}\left(\frac{z - z_0}{2}\right)\right) + \sum_{k=1}^{\infty} a_k \exp(ikz),$$

for P a complex-valued polynomials and $|a_n|^{1/n}$ is bounded; equivalently, $f(z) = P(\mathrm{cotan}(z)) + g(z)$, for $g(z)$ holomorphic on all of $\mathbb{H}$. Hint: use an inductive argument.

2. Let $\Omega := \mathbb{D} \setminus \{0\}$ be the domain given by removing the origin from the disk. *All parts of this question can be done with only material from the first half of class (no conformal results necessary).*

   (a) (5 points) Let Ω' be a bounded open domain. Let $f : \Omega \to \Omega'$ be a holomorphic function. Show that $\lim_{z \to 0} f(z)$ exists.

   (b) (10 points) Give a counterexample when Ω' is not bounded. Show that moreover, there exists a holomorphic function $F : \Omega \to \mathbb{C}$ such that the limit $\lim_{z \to 0} f(z)$ does not exist, and it is also not true that $\lim_{z \to 0} |f(z)| = \infty$. (This is equivalent to the function $P_N^{-1} \circ F : \Omega \to S^2$ having no limit point as $z \to 0$.)

   (c) (10 points) Assume $f : \Omega \to \Omega$ is a holomorphic *bijection* from Ω to itself (equivalently, a conformal mapping). Show that the function

$$f_{ex}(z) := \begin{cases} 0, & z = 0 \\ f(z), & z \neq 0 \end{cases}$$

[1] the original problem contained a typo. You can alternatively assume f is defined on an open neighborhood $\mathbb{H}_{-\epsilon} := \{z \mid \mathrm{Im}(z) > -\epsilon\}$ of the *closed* half-plane and then show $\sum |a_n|$ converges.

[2] the typo also affected this part; here you can alternatively show separately that $\sum |a_n| r^n < \infty$ implies f extends to $\mathrm{Im}(z) > -\log(r)$ and that f being holomorphic in a domain containing $\{z \mid \mathrm{Im}(z) > -\log(s)\}$ for $s > r$ implies $\sum |a_n| r^n < \infty$.

(the "extension" of f) is a conformal mapping from $\mathbb{D}$ to $\mathbb{D}$.

Hint: By (a), the limit $\lim_{z\to 0} f(z)$ exists; call it w. It remains to show that this limit $w = 0$, instead of some other point of Ω or its boundary. One way to do this is to first use the Cauchy inequalities to prove $w \in \mathbb{D}$ (i.e., $|w| < 1$), then deduce a contradiction by a limit/continuity argument that if $w = f(z_0)$ for some $z_0 \in \Omega$, then f itself cannot be a bijection.

3. Let $r \in (0, 1)$ be a real number between 0 and 1. Let $\Omega = \mathbb{D} \setminus \{r, -r\}$. By arguments similar to 2c (which you do not need to understand to solve this problem), any conformal mapping $\Omega \to \Omega$ is the restriction to $\mathbb{D} \setminus \{r, -r\}$ of a mapping $\mathbb{D} \to \mathbb{D}$, which moreover maps the set $\{r, -r\}$ to itself. You may use this fact in the following.
   (a) (5 points) Construct two distinct conformal mappings $f : \Omega \to \Omega$.
   (b) (5 points) Construct a conformal mapping $g : \Omega \to \mathbb{D} \setminus \{0, \frac{2r}{1+r^2}\}$.
   (c) (10 points) Show that if $r \neq s$ are two different numbers on $(0, 1)$, there can be no conformal mapping from $\Omega = \mathbb{D} \setminus \{\pm r\}$ to $\Omega' := \mathbb{D} \setminus \{\pm s\}$. Deduce that the sets of the form $\mathbb{D} \setminus \{\pm r\}$ give an infinite (indeed, an uncountable) collection of domains which are not conformally equivalent. You may use that any conformal mapping $g : \mathbb{D} \setminus \{-r, r\} \to \mathbb{D} \setminus \{-s, s\}$ is the restriction of a conformal mapping $g_{ex} : \mathbb{D} \to \mathbb{D}$ which takes the set $\{\pm r\}$ to $\{\pm s\}$.

4. Let $\mathbb{H} \subset \mathbb{C}$ be the upper half-plane. Let $f : \mathbb{H} \to \mathbb{R}^3$ be the map (i.e., function) written in polar coordinates as

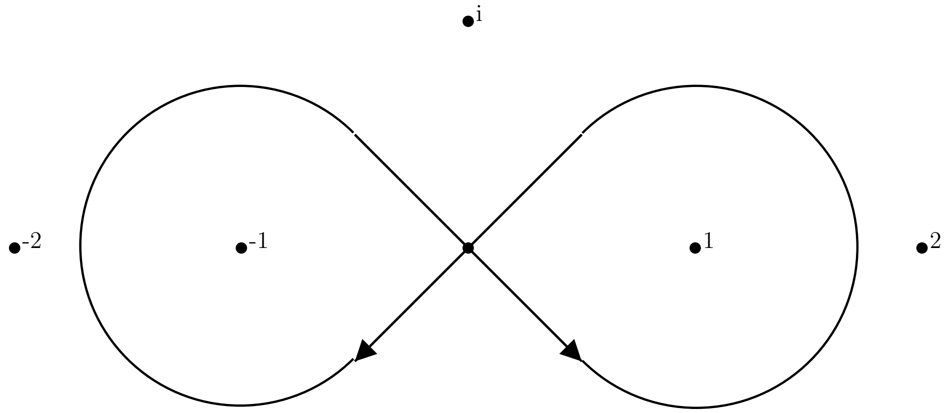
$$f : r\exp(i\theta) \mapsto \left(\frac{r}{2}\cos 2\theta,\ \frac{r}{2}\sin 2\theta,\ \frac{\sqrt{3}r}{2}\right).$$

   (a) (10 points) Show that f is a (not necessarily bijective) conformal function. (It may help to know that if $z_0 = r\exp(i\theta)$, the tangent vectors to the two curves $\gamma_1(t) = (r+t)\cdot\exp(i\theta)$ and $\gamma_2(t) = r\cdot\exp\left(i(\theta + \frac{t}{r})\right)$ through z_0 form an orthonormal basis for the vector space $\mathbb{C}$, with tangent vectors $\dot{\gamma}_1(0) = \exp(i\theta)$, $\dot{\gamma}_2(0) = i\exp(i\theta)$.)
   (b) (10 points) Let $K_+ \subset \mathbb{R}^3$ be the "open cone"

$$K := \{(x, y, z) \mid z > 0, z = \sqrt{3}|(x, y)|.\}$$

   ("Open" in the sense that we have removed the corner $(0, 0, 0)$ in the definition of K.) Construct a conformal equivalence between K and $\mathbb{C}^*$ (here $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.) Hint: use (a). The function $z \mapsto z^2$ is your friend, though the function $z \mapsto \sqrt{z}$ would be even more useful if it were defined...)

5. Let γ be the curve given by the figure 8, as drawn in the following picture:



(In formulas: γ is the union of the segment $\gamma_1(t) = (1-i)t$ for $-1/2 \le t \le 1/2$, the arc $\gamma_2(t) = 1 + \frac{\sqrt{2}}{2}\exp(-it)$ for $-3\pi/4 \le t \le 3\pi/4$, the segment $\gamma_3(t) = (-1-i)t$ for $-1/2 < t < 1/2$ and the arc $\gamma_4(t) = -1 + \frac{\sqrt{2}}{2}\exp(it)$ for $-3\pi/4 \le t \le 3\pi/4$.)

(a) (10 points) Prove carefully that if f is a meromorphic function with no poles on γ which is odd (i.e. $f(-z) = -f(z)$) then $\int_\gamma f(z)dz = 0$.

(b) (10 points) Compute

$$\int_\gamma \frac{\exp(\pi i z)}{\cos^2(\pi z)} dz.$$

6. Let $f(z)$ be a meromorphic function such that $f(z+1) = f(z+i) = f(z)$, such that $f(z)$ has no zeroes or singularities with either $\mathrm{Re}(z)$ or $\mathrm{Im}(z)$ an integer. Let $S = \overline{0,1} \cup \overline{1,1+i} \cup \overline{1+i,i} \cup \overline{i,0}$ be the unit square with the four vertices $0, 1, 1+i, i$.

(a) (10 points) Show that $\int_S f(z) = 0$.

(b) (10 points) Deduce that the number of poles (counted with multiplicity) of f in the interior of S is equal to the number of zeroes of f on the interior of S (hint: consider $F(z) = \frac{f'}{f}$).