Math 141 Quiz 1 September 12, 2019

Additional sheets available (write your name on any additional sheets!!)

Name: ______________________________

1. Let $S^1 \subset \mathbb{R}^2$ be the circle, i.e. the set of points satisfying the equation $x^2+y^2=1$. Let $X := S^1 \times S^1 \subset \mathbb{R}^4$.
   (a) Write down an expression without product notation for the set $X \subset \mathbb{R}^4$ as above (with coordinates x, y, z, t).

   $\{(x,y,z,t) \mid x^2+y^2 = z^2+t^2 = 1\}$

   (b) Write down a collection of charts (or if you prefer, coordinate systems) which cover X. (It is enough to write down the subsets of $\mathbb{R}^n$ and maps involved, no need to check they are smooth, etc.)

   Let $f : \mathbb{R}^2 \to S^1 \times S^1$ be given by $f(s,t) := (\cos(s), \sin(s), \cos(t), \sin(t))$

   Let $U_1 = (0,2\pi) \times (0,2\pi)$, $U_2 = (0,2\pi) \times (-\pi,\pi)$, $U_3 = (-\pi,\pi) \times (0,2\pi)$, $U_3 = (-\pi,\pi) \times (-\pi,\pi)$
   Then $f|_{U_1}, f|_{U_2}, \dots, f|_{U_4}$ are charts which cover X.

   (c) Let S^3_r be the 3-dimensional sphere of radius r, with equation $x^2+y^2+z^2+t^2=r^2$. Find (with proof) r such that

   $$X \subset S^3_r.$$

   Is X equal to S^3_r for this value of r? Why or why not?

   $(x,y,z,t) \in S^1 \times S^1 \Rightarrow x^2+y^2 = z^2+t^2 = 1 \Rightarrow x^2+y^2+z^2+t^2 = 2 = \sqrt{2}^2$. So $X \subset S^3_{\sqrt{2}}$. Since $(1,1,0,0) \in S^3_{\sqrt{2}}$ but not in $S^1 \times S^1$, not equal. [Also: $\dim S^1 \times S^1 = 2$, $\dim S^3_{\sqrt{2}} = 3$

2. Let $f : \mathbb{R}^1 \to \mathbb{R}^2$ be defined as $f(\theta) = (\cos(\theta), \sin(\theta))$. Let $X = f(\mathbb{R})$ be the *image* of f.
   (a) What is X? (Give a different expression, or a name.)

   $X = S^1$

   (b) What is the preimage $f^{-1}(p)$ of the point $p = (\sqrt{3}/2, 1/2)$?

   $f^{-1}(p) = \{\frac{\pi}{3} + 2\pi k \mid k \in \mathbb{Z}\}$

   (c) Let $I = (0, 2\pi)$. Prove that $f(I)$ open in X but not equal to X. Show that the closure $\overline{f(I)} \subset \mathbb{R}^2$ is X.

   $f(I) = S^1 \setminus \{(1,0)\}$. Since $\{(1,0)\}$ is closed in $\mathbb{R}^2$, know $\{(1,0)\} = \{(1,0)\} \cap S^1$ closed in $S^1 \Rightarrow f(I)$ open. Since $\sin$, $\cos$ are continuous f is continuous $\Rightarrow \lim_{n\to\infty} f(\pi/n) = f(0) = (1,0)$. Since $f(\pi/n) \in f(I)\ \forall n$, deduce $f(0) \in \overline{f(I)} \Rightarrow \overline{f(I)} = S^1$

3. True or false? If you have time, give a very short ($\sim$ one sentence) explanation or counterexample.

   (a) If f is a diffeomorphism then it is a homeomorphism.

   True : smooth $\Rightarrow$ continuous.

   (b) A subset of a compact topological space is compact.

   False : $(0,1) \subset [0,1]$ is a counterexample

   (c) The image of the linear operator defined by the matrix

   $$\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$

   is a manifold.

   True : The image is $\{ r \cdot \binom{1}{2} \mid r \in \mathbb{R} \}$, a line. This is a one-dim manifold (chart given for example by $\varphi(r) = r \cdot \binom{1}{2}$, works for all points)

   (d) The sphere $S^2 \subset \mathbb{R}^3$ contains a subset diffeomorphic to $\mathbb{R}^2$.

   True. $S^2 \setminus (0,0,1) \underset{\text{diffeo}}{\cong} \mathbb{R}^2$ via stereographic projection.

   Or: S^2 is a manifold $\Rightarrow$ has subset diffeo to open ball in $\mathbb{R}^2$ & open balls $\subset \mathbb{R}^2$ are diffeo to all of $\mathbb{R}^2$.