Math 141 Homework 6. Due 10/22
Transversality and homotopy

Reminder: if you are interested in having your solutions used for the answer key, please email me your TeX file with any style files you use.

"GPn" denotes Guillemin-Pollack, chapter 1, section n.

1. GP5, 1 and 2 (careful proof not needed for 2, drawings encouraged.)

2. GP6, 1 and 2. You may use the function ρ in the hint for problem 1 as given (in fact, you've constructed a function of this sort in a previous homework).

No need to read all of chapter 6 for these problems: it is sufficient to know the following definition.

Definition. Two smooth maps $f, g : X \to Y$ are said to be *homotopic* if for the interval $I = [0,1]$ there exists a smooth map $h : X \times I \to Y$ such that $h(p,0) = f(p)$ and $h(p,1) = g(p)$ for $p \in X$. In other words, f and g are homotopic if there is a family of smooth functions $h_t : X \to Y$ parametrized by $t \in [0,1]$ which interpolates between $h_0 = f$ and $h_1 = g$, in such a way that for an $p \in X$, the values $h_t(p)$ depend smoothly on t.

3, extra credit. (a) Let $H_+ \subset \mathbb{R}^2$ be the open upper half-plane, given by $H_+ = \{(x,y) \mid y > 0\}$ and $H_- \subset \mathbb{R}^2$ the lower half-plane given by $H_- = \{(x,y) \mid t < 0\}$. Let $L = \{(x,0), x \in \mathbb{R}\}$ be the x-axis. Let $I = [0,1]$ and $f : I \to \mathbb{R}^2$ is a smooth embedding which maps the endpoints to $\mathbb{R} \setminus L$ (so either to H_- or to H_+) and is transversal to L (by this I mean that the restriction $f\mid_{\mathring{I}}$ to the interior, $\mathring{I} = (0,1) \subset I$ is transvesral, as I is not a manifold while $\mathring{I}$ is; since the endpoints aren't allowed to map to L, this is a technicality).
(a) Show that the intersection $f(I) \cap L$ has finitely many points. (Hint: a compact zero-dimensional manifold is a finite set of points.)
(b) Assume $f(I)$ intersects with L at finitely many points. Show that the number of intersections of $f(I)$ with L is even if $f(0), f(1)$ are both in the same half-plane and odd if they are in different half-planes. (Hint: show that intersection points represent places where $f(t)$ transitions from H_+ to H_- or vice versa.)
(c) Show that if $g : S^1 \to \mathbb{R}^2$ is an embedding transversal to L then it has a finite and even number of intersection points with L (hint: split the circle into

two line segments).
(d) Give a counterexample to (c) when g is not assumed to be an embedding (but is still assumed to be transversal to L) — a picture will suffice. However, give "corrected" statements for (b), (c) that hold when f, g are not assumed to be embeddings (no need to prove these: the proof will be almost exactly the same).

4. (Not for credit). How hard did you find this homework set? How much time, approximately, did it take you to do the problems?