Math 141 Homework 4. Due 10/1

"GPn" denotes Guillemin-Pollack, chapter 1, section n.

1. GP4, 9-10

2. GP4, 11

3. Recall that a Lie group is a *subgroup* of some GL_n which is a manifold. Suppose $f : G \to H$ is a map of Lie groups which is both a smooth map and a *homomorphism*, i.e. such that $f(g_1 \cdot g_2) = f(g_1) \cdot f(g_2)$. Show that f is a (submersion, immersion, local diffeomorphism) if and only if $d_e f$ (the differential at the identity) is, respectively, a (surjective, injective, bijective) linear map. Give an example of a map of connected Lie groups which is a local diffeomorphism but not a diffeomorphism. (Hint: start by showing that if G is *abelian*, such as $SO(2)$, then the map $g \mapsto g^2$ is a homomorphism. More interesting non-abelian examples also exist.)

4. (This may be easier on a piece of paper!)

(a) Draw or write down two maps $X \to \mathbb{R}$ with different but finite numbers of critical values, for two manifolds X diffeomorphic to the sphere. Draw a picture of the map and the preimage of each critical value, and compute the Euler characteristic of each resulting one-dimensional manifold (defined as $V - E$). Check that in each case the Euler characteristics sum up to 2, which is the Euler characteristic of S^2.

(b) Do the same thing with X diffeomorphic to the torus, $S^1 \times S^1$.

Hint for drawing maps to $\mathbb{R}$: start by drawing a shape in $\mathbb{R}^3$ which can be smoothly deformed to a sphere or a torus, then use projection to the z-axis to define the map.

5, extra credit. (a) Show that if $G \subset GL_n$ is a Lie group then the tangent space T_eG is closed under the *commutator* operation, $[u, v] := u \cdot v - v \cdot u$.

(b) Assume that $A \subset \mathrm{Mat}_{n\times n}$ is a vector subspace which is closed under the commutation operation $[u, v] = u \cdot v - v \cdot u$ (such a subspace is called a *Lie algebra*, as opposed to a Lie group). Check that for any $u, v \in A$ the series $\log(\exp(\epsilon u) \cdot \exp(\epsilon v))$ is a matrix-valued power series in ϵ whose terms **to order** ϵ^2 (ignoring ϵ^3 and higher) are in A. (Here the function log on matrices is defined by the usual power series: $\log(1 + M) = \sum_{k=1}^{\infty} \frac{(-1)^{n-1}}{n} M^n$,) and is (when defined) an inverse to the exp function. Show that the condition that A is closed under the $[\,,\,]$ operation is necessary: i.e., give an example of a subspace $A \subset \mathrm{Mat}_{2\times 2}$ and $u, v \in A$ such that $\log(\exp(\epsilon u)\exp(\epsilon v))$ has a coefficient not in A.

In fact, if A is a Lie algebra (i.e., closed under $[,]$) then the whole power series for $\log(\exp(\epsilon u)\exp(\epsilon v))$ has coefficients in A. From this one can see that if $A \subset \mathrm{Mat}_{n\times n}$ is a Lie algebra and $v, w \in A$ vectors then there exists some $u \in A$ such that $\exp(v)\exp(w) = \exp(u)$, i.e. the subset $\exp(A) \subset \mathrm{GL}_n$ is a subgroup! It might not be a manifold, but it turns out that A is isomorphic to some Lie algebra $A' \in \mathrm{Mat}_{N\times N}$ (in a possibly larger ambient space) such that $\exp(A')$ is a Lie group. In other words, (a) and (b) together imply that the tangent space to a Lie group is a Lie algebra, and every Lie algebra can be "exponentiated" to a Lie group! This means that Lie groups and Lie algebras capture *almost* the same information, the foundational insight of Lie theory.[1]

6. (Not for credit). How hard did you find this homework set? How much time, approximately, did it take you to do the problems?

[1] In fact, the Lie group contains a little more information as the map $\exp : T_e(G) \to G$ is a local diffeomorphism but might not be a diffeomorphism.