Math 141 Homework 3. Due 9/24

"GPn" denotes Guillemin-Pollack, chapter 1, section n.

1. GP2, 8

2. GP2, 10, 11. You may use any and all parts of GP2, 9.

3. GP2, 12. Ignore the words "and conversely" (you already proved the other direction in a previous part).

4. GP3, 7

5. Extra credit. (Based on previous problem.) For $X \subset \mathbb{R}^N$ any subset and $p \in X$ any point, let $\mathrm{Curve}_p(X)$ be the set of smooth curves $\mathrm{Curve}_p(X) := \{c : [0, \epsilon) \to X \mid \epsilon > 0, c(0) = p\}$. Define an equivalence relation on $\mathrm{Curve}_p(X)$ as follows: for $c : [0, \epsilon) \to X$ and $c' : [0, \epsilon') \to X$ two elements of $\mathrm{Curve}_p(X)$, we say $c \sim c'$ if when restricted to $[0, \min(\epsilon, \epsilon'))$, the function $c - c' : I \to \mathbb{R}^N$ has zero derivative at 0.

Define

$$T_p^{int}(X) := \mathrm{Curve}_p(X)/ \sim$$

to be the set of equivalence classes of curves under $\sim$ (here the superscript "int" stands for intrinsic).

(a) Prove that, for X a manifold, two curves c, c' are equivalent as above if and only if $d_0(c) = d_0(c')$. Deduce that $T_p^{int}(X)$ is in bijection with $T_p(X)$.

(b) If $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$ are spaces and $f : X \to Y$ is a smooth map between spaces (not necessarily manifolds) with $f(p) = q$, we can take any curve into X and "promote" it to a curve into Y. In other words, we have a natural map

$$d_{\mathrm{Curve}} f : \mathrm{Curve}_p(X) \to \mathrm{Curve}_q(Y)$$

given by $d_{\mathrm{Curve}} f : c \mapsto f \circ c$. Show, without using other definitions, that d_{Curve} induces a well-defined map $d^{int} : T_p(X) \to T_q(Y)$.

(c) Show that for the coordinate cross $X = \{x, y \mid xy = 0\}$, the "intrinsic" tangent space $T_p^{int}(X)$ at $p = (0, 0)$ is not a vector space. Deduce another proof that X is not diffeomorphic to a manifold.

4.

5.