Math 185 Homework 1. Due Friday 1/31 (later homeworks due Wednesday)

1. Define $\exp(iy) := \cos(y) + i\sin(y)$.
a. Prove, using trigonometry, that $\exp(iy + iy') = \exp(iy) \cdot \exp(iy')$ for $y, y' \in \mathbb{R}$ two real numbers.
b. Prove directly (using Taylor series for sin and cos) that

$$\exp(iy) = \sum_{n=1}^{\infty} \frac{(iy)^n}{n!},$$

where $n!$ denotes the factorial of n. Hint: you may use the fact that an infinite sum of complex numbers $\sum a_n$ converges if and only if $\sum \mathrm{Re}(a_n)$ and $\sum \mathrm{Im}(a_n)$ both converge and if it converges, $\sum a_n = \sum \mathrm{Re}(a_n) + i \sum \mathrm{Im}(a_n)$. Now apply this to $a_n = \frac{(iy)^n}{n!}$.

2. This and the following exercise are meant to help develop your thinking about complex numbers. They do not follow the book: you will need to think a bit on your own in order to solve these. For a positive real number $r \in \mathbb{R}$, define

$$C_r := \{z \mid \quad |z| = r\}$$

to be the circle of radius r around 0.

Let $\mathbb{G} = \{x + iy \mid x, y \in \mathbb{Z}\}$ (called the set of "Gaussian numbers") be the set of complex numbers with integer real and imaginary part.
a. Prove that the product $z \cdot z'$ of two elements $z, z' \in \mathbb{G}$ is again in $\mathbb{G}$.
b. Prove that $\mathbb{G} \cap C_1 = \{\pm 1, \pm i\}$. In other words, the only elements $z \in \mathbb{G}$ with $|z| = 1$ are the four distinct powers or i.

From now on, we write $U_4 := \{\pm 1, \pm i\}$ (here U_4 stands for "fourth roots of unity").

c. Prove that if $|z| = r$ then $|uz| = r$ for $u \in U_4$ and $|\bar{z}| = r$. Let $C_r \subset \mathbb{C}$ be the circle of radius r, given by $C_r = \{z \in \mathbb{C} \mid |z| = r\}$. Show that the number of points $|C_r|$ is finite and has number of elements divisible by 4[1]. (Hint: the set $\{\pm 1, \pm i\}$ has four elements).

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[1] if $C_r \cap \mathbb{G}$ is empty, it has 0 elemnts, which is divisible by 4.

d. Show that if (for two numbers $r, s \in \mathbb{R}$), the circles C_r and C_s both contain a Gaussian number then the circle C_{rs} also contains a Gaussian number. Deduce that if m, n are integers which can be expressed as the sum of two squares then mn can be as well (hint: show that m is the sum of two squares if and only if $C_{\sqrt{m}}$ contains a Gaussian number).
e. Find all Gaussian numbers of length $\sqrt{5}$, i.e. all numbers in $C_{\sqrt{5}} \cap \mathbb{G}$. Sketch them (or draw them on graph paper.) Connect pairs of numbers which are related by multiplication by $\pm i$. (This should split your numbers into "squares").

3. Now we do the same thing for the ring of *Eisenstein integers.* Define the set of Eisenstein integers $\mathbb{E}$ to be the set of integers $\mathbb{E} := \{\frac{a+b\sqrt{3}i}{2} \mid a \equiv b \mod 2$. So for example, $-5\sqrt{3}i \in \mathbb{E}$ and $3 - \sqrt{3}i \in \mathbb{E}$ bue $1 + \frac{\sqrt{3}}{2}$ is not in $\mathbb{E}$.
a. Draw a (sketch) of the Eisenstein integer lattice. (You should get something with hezagonal symmetry!) Show that the set of Eisenstein integers is closed under multiplication, so if $z, z' \in \mathbb{E}$, then so is $z \cdot z'$.
b. Let $\zeta := \exp(\frac{2\pi i}{6})$, also known as "the primitive sixth root of unity". (The Greek letter ζ is pronounced "zeta" and written "\zeta" in LaTeX). Show that $\zeta \in \mathbb{E}$ (in fact, you can observe that $\mathbb{E} = \{a + b\zeta \mid a, b \in \mathbb{Z}\}$). Show that $\zeta^6 = 1$, that $-\zeta = \zeta^4$ and $\bar{\zeta} = \zeta^{-1}$.
c. Show that $C_1 \cap \mathbb{E} = \{1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^5\}$ is the set of the six distinct powers of ζ. (The notation C_r is, as before, the circle of radius r.)

From now on, we write $U_6 := \{\zeta^k, 0 \leq k \leq 5\}$ for the set of unit Eisenstein numbers (here U_6 stands for "sixth roots of unity").

d. Show that if $z \in C_r$ (equivalently, $|z| = r$) then $\zeta^n z$ and $\bar{z}$ are also in C_r. Deduce that the set of Eisenstein integers in the circle C_r has number of elements divisible by 6.
e. Find and draw twelve elements in $C_{\sqrt{7}} \cap \mathbb{E}$ (these are in fact all the elements of $\mathbb{E}$ of length $\sqrt{7}$). Connect by a segment pairs of elements related by multiplication by ζ. (You should get two hexagons each consisting of groups of U_6-multiples!)

4. Fix a positive integer n. Let $z_{a,b} \in \mathbb{C}$ be an array of numbers indexed by pairs of integers a, b with $0 \leq a \leq n$ and $0 \leq b \leq n$ (you can think of this as an $n+1$ by $n+1$ square matrix, but thinking of $z_{a,b}$ as being in the point (a, b) of the plane rather than (b, a) as would be the case for matrix notation). Let $h_{a,b} := z_{a+1,b} - z_{a,b}$ for $0 \leq a \leq n-1, 0 \leq b \leq n$ be the matrix of horizontal differences (notice that $z_{a+1,b}$ only makes sense for $a \leq n-1$). Similarly, let $v_{a,b} := z_{a,b+1} - z_{a,b}$ for $0 \leq a \leq n$ and $0 \leq b \leq n-1$ be the matrix of vertical differences.
a. Show that for any pair of indices $a, b \in \{0, \ldots, n-1\}$ we have

$$v_{a,b} - v_{a+1,b} = h_{a,b} - h_{a,b+1} \tag{1}$$

It is helpful to think of the difference $v_{a,b}$ as corresponding to the vertical edge between the points (a, b) and $(a, b+1)$ and similarly for $h_{a,b}$ on a horizontal edge. This question is asking you to prove an identity about the numbers written on the edges of the little square connecting the four vertices $(a, b), (a+1, b), (a+1, b+1)$ and $(a, b+1)$.

b. Conversely, show that if we have collections of numbers $v_{a,b}$ (for $a \leq n, b \leq n-1$) and $h_{a,b}$ (for $a \leq n-1, b \leq n$) as above which satisfy equation (1) then there exists a collection of $z_{a,b}$ with $h_{a,b} = z_{a+1,b} - z_{a,b}$ and $v_{a,b} = z_{a,b+1} - z_{a,b}$, and that any two possibilities for the numbers $z_{a,b}$ differ from each other by a constant.

Hint: Assume that $z_{0,0}$ is some constant number $c \in \mathbb{C}$. By considering the differences between consecutive pairs in the path $z_{0,0} \to z_{1,0} \to \cdots \to z_{a,0} \to z_{a,1} \to z_{a,2} \to \cdots \to z_{a,b}$, write a expression for $z_{a,b}$ in terms of $v_{j,k}$ and $h_{j,k}$. Now check that $h_{j,k}$ and $v_{j,k}$ are indeed the differences.

c. Let $\lambda_h, \lambda_v \in \mathbb{C}$ be two arbitrary complex numbers. Define arrays

$$h_{a,b} := \lambda_h \cdot (a + bi)$$

and

$$v_{a,b} := \lambda_v \cdot (a + bi).$$

Show that equation (1) is satisfied (so these particular choices $h_{a,b}, v_{a,b}$ are indeed differences) of and only if $\lambda_v = i \cdot \lambda_h$.

Notice the similarity between the condition $\lambda_v = i \cdot \lambda_h$ and complex differentiability. A continuous version of this type of argument (with $z_{a,b} := f(\frac{a+bi}{n})$, and with n approaching ∞) is useful for proving integration and differentiation formulas for holomorphic functions.