Math 141, Prof. Vaintrob
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Homework 9

GP 2.1

Exercise 2

We have that ∂X must map to ∂Y, for if it mapped to an interior point we could compose f on the left with a coordinate system to $\mathbb{R}^k$ and on the right with a parametrization from a boundary point of H^k, contradicting Exercise 1 (any particular neighborhoods we take are obviously diffeomorphic to neighborhoods of 0 in $\mathbb{R}^k$ and H^k respectively).

The same reasoning applies to ∂Y and f^{-1}. Thus ∂f is a bijection of the boundaries.

Smoothness of ∂f and its inverse follow just because a smooth extension of f with be a smooth extension of ∂f.

Exercise 3

Suppose the filled-in unit square S is a manifold with boundary. There are obvious diffeomorphisms of neighborhoods of side points which do not contain any corners to H^2. Therefore they are in ∂S. But then the corners must also be in ∂S, for any neighborhoods of them also include some side points, which cannot be in the image of a parametrization by an open of $\mathbb{R}^2$, because we've just shown they're boundary.

So ∂S is the empty unit square. But the boundary has to be a manifold without boundary, and we know the empty unit square is not.

Indeed suppose we had a parametrization of a neighborhood of a corner in the empty unit square, i.e. a smooth function $f : \mathbb{R} \to \mathbb{R}^2$ with nonzero derivative and image a neighborhood of one of the corners. This can't be, because approaching f^{-1}(the corner) from the left the derivative is nonzero and always parallel to the side we're on, and approaching from the right it's parallel to the other side, which is perpendicular to the first one. This isn't a continuous vector-valued function. $\square$

Exercise 4

We note that zero is a regular value of the function $f(x, y, z) = a + z^2 - x^2 - y^2$ from $\mathbb{R}^3$ to $\mathbb{R}$. Indeed the differential is given by $\begin{bmatrix} -2x \\ -2y \\ 2z \end{bmatrix}$, which is only zero at $(0, 0, 0)$. Therefore because f is real-valued, $(0, 0, 0)$ is the only point where it fails to be a submersion, and this point is not in the preimage of zero.

Therefore by the Lemma at the end of the proof of the Preimage Theorem for manifolds with boundary (p. 62), the set where we have $f(x, y, z) \geq 0$ is a manifold with boundary. This is exactly the solid hyperboloid.

Exercise 6

We know that that the boundary of any such construction (sort of assuming it yields some 2-manifold with boundary) is a compact one-manifold without boundary. Therefore by the classification of one-manifolds it's a union of copies of S^1. clearly there's two connected components if we have an even number of twists, and one if we have an odd number; this gives the desired conclusion. I will not put a box after this proof.

Exercise 7

Let ϕ and ψ be two parametrizations from opens of H^k, both taking zero to x, such that $(d\phi)_0(H^k) \neq (d\psi)_0(H^k)$. Then $g = \phi^{-1} \circ \psi$ is a diffeomorphism between some sufficiently small neighborhoods of zero in H^k, which fixes 0. But dg_0 does not take H^k onto itself. (This follows from $(d\phi)_0(H^k) \neq (d\psi)_0(H^k)$, and the fact that $dg_0 = d(\phi^{-1})_x \circ (d\phi)_0 = (d\phi_0)^{-1} \circ (d\phi)_0$.) We seek a contradiction from the existence of such a g.

Indeed, as g takes zero to zero, it is equal to its differential at zero up to first order. Take $x \in H^k$ such that $dg(x) \notin H^k$; $dg(x)$ has a finite nonzero distance from H^k. As dg is linear, this distance is proportional to the magnitude of x. Therefore it goes to zero linearly as x does. But $g(x)$ must go to $dg(x)$ quadratically as $x \to 0$; this implies that for sufficiently small multiples of x, $g(x)$ has nonzero distance from H^k. But that's a contradiction as g is supposed to map into H^k. □

GP 2.2

Exercise 4

Just take the affine linear map $\mathbf{x} \mapsto \frac{1}{2}[\mathbf{x} + (\sqrt{a}, 0, 0, \dots)]$. By the triangle inequality it's a map from the open ball to itself, as points in the image have magnitude less than 1; algebra shows that the only fixed point we can have is $(\sqrt{a}, 0, 0, \dots)$, which is on the boundary and so isn't included. □