Math 141 Homework 8.

November 17, 2019

1. (GP7, 3). (Hint: use the fact from class that for any collection of open neighborhoods N_q for each $q \in Z$, there is a countable collection $N_{q_1}, N_{q_2}, \ldots$ covering Z.

Proof. Assume that $X \subset Z$ is a submanifold and $\dim Z = k < n \dim X$. From the hint, we know that Z is covered by countably many n'hoods. Thus, if it can be shown that each n'hood of Z has measure zero, then it follows from the fact in GP (pg. 40) that a countable union of measure zero sets has measure zero. Thus fix $p \in Z$. If we consider a n'hood of p, N_p then there is chart that diffeomorphically maps $\mathbb{R}^k \to N_p$. Thus N_p is a k dimensional subset X. This leaves $n - k$ dimensions that can be constructed to show that N_p in fact has zero volume inside of X. It suffices to look at the firect product with the correct copies of a small interval $I_\epsilon = [-(\epsilon/2)^{\frac{1}{n-k}}/C, (\epsilon/2)^{\frac{1}{n-k}}/C]$:

$$N_p \times I_e^{n-k}$$

With the constant C, it is then possible to shrink the volume of N_p in X down to an epsilon of arbitrary size. This that any n'hood of a point p has measure zero. Thus a countable union of n'hoods must have measure zero. This proves that Z has measure zero. □

2. (GP7, 4)

Proof. From the same fact on pg. 40 of GP, and the fact that the rational numbers are a countable subset of $\mathbb{R}$, it suffices to show that each singleton $\{q\}$ has measure zero. Since $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ this would imply that the rationals, a dense set, have measure zero in $\mathbb{R}$. This is easy to show since the singleton set $\{q\}$ can be covered by the interval $[q - \epsilon, q + \epsilon]$. Thus we can restrict the volume of a singleton in one dimension to an arbitrarily small value. This proves the rationals have measure zero inside the reals. □

3. (GP7, 18) It can show that on a compact manifold X, a Morse function is stable under (sufficiently small) perturbation if it is stable to perturbation on each n'hood. Since $X = \bigcup_{p \in X} N_p$ then we can use the minimum "stable t" to show that f stays Morse on all of X. Each N_p is a n'hood of a compact set X, and thus from the result of Ex. 17 (which we were graciously granted permission to use) it follows that a family of homotopic function remains Morse for sufficiently small t. Thus f must be stable on all of X if we choose the smallest of these perturbations that keeps the homotopies f_t Morse.

4. (GP8, 6)

Proof. We can use the actual definition of a vector field $\vec{v} : X \to \mathbb{R}^N$, where $\vec{v}(x) \in T_x(X)$, to define a new function that can be used to answer this question:

$$v : X \to T(X)$$

$$v = id_X \times \vec{v}$$

$$v : x \mapsto (x, \vec{v}(x))$$

Now it is easy to check that $p \circ v = id_X$. Take $x \in X$. Then $p \circ v(x) = p(v(x)) = p(x, \vec{v}(x)) = x$.

□

(GP8, 8)

Proof. If S^k has a non vanishing vector field, then at each $p \in X$ you can pick out a nonzero vector. Call this vector $\vec{v} \in p^\perp$. Let P be the plane defined by the span of $\langle p, \vec{v} \rangle$. This gives an equator path from p to the antipodal point and back around to p. Since each of the vectors are nonzero, one can define the unit vector in the direction of $\vec{v}$: $\hat{v} = \frac{\vec{v}}{|\vec{v}|}$. Now we have everything needed to define a homotopy from the identity to the antipodal map:

$$f_t = \cos(\pi t)p + \sin(\pi t)\hat{v}$$

□

Since $\hat{v} \in p^\perp$, it follows that $|f_t(x)| = 1$, i.e. that $f_t(x) \in S^k$.