Math 141 Homework 7

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1.(G.P 5.4) Let X and X be transversal submanifolds of Y. Prove that if $y \in X \cap Z$, then

$$T_y(X \cap Z) = T_y(X) \cap T_y(Z)$$

Solution. Let $dim(Y) = n$. Suppose $y \in X \cap Z$, since $X \cap Z \subset X$, we have $T_y(X \cap Z) \subset T_y(X)$. Similarly, we have $T_y(X \cap Z) \subset T_y(Z)$. As a result, $T_y(X \cap Z) \subset T_y(X) \cap T_y(Z)$. Now since X, Z are intersecting manifolds, we have $codim(X \cap Z) = codim(X) + codim(Z)$. In other words, $dim(X \cap Z) = dim(Y) - codim(X) - codim(Z) = dim(Y) - (dim(Y) - dim(X)) - (dim(Y) - dim(Z)) = dim(X) + dim(Z) - n$, which implies that

$$dim(T_y(X \cap Z)) = dim(T_y(X)) + dim(T_y(Z)) - n$$

.
On the other hand, by linear algrbra, $dim(T_y(X) \cap T_y(Z)) = dim(T_y(X)) + dim(T_y(Z)) - dim(T_y(X) + T_y(Z))$ Since $X \pitchfork Z$, we have that $dim(T_y(X) + T_y(Z)) = dim(T_y(Y)) = n$. Therefore, we have

$$dim(T_y(X) \cap T_y(Z)) = dim(T_y(X)) + dim(T_y(Z)) - n$$

.
Hence, $dim(T_y(X \cap Z)) = dim(T_y(X) \cap T_y(Z))$ and since $T_y(X \cap Z) \subset T_y(X) \cap T_y(Z)$, we have that $T_y(X \cap Z) = T_y(X) \cap T_y(Z)$.

(G.P 5.6) Suppose that X and Z do not intersect transversally in Y. May $X \cap Z$ still be a manifold? If so, must its codimension still be $codimX + codimZ$? (Can it be?) Answer with drawings.

Solution. (Drawings see appendix) Yes, $X \cap Z$ can still be a manifold. Consider let X be the x-axis in R^2, Z be the x-axis in R^2. Both X and Y are manifolds and the intersection $X \cap Z$ is the x-axis, which is still a manifold.

The codimension NOT necessarily $codimX + codimZ$. Let $dim(Y) = n$. Let X, Z be the two orthogonal axes in R^3 respectively. Then, $codim(X \cap Z) = 3$ while $codim(X) + codim(Z) = 2 + 2 = 4$.

But the codimenstion can still be $codim(X) + codim(Z)$. Let X, Z be the two axes in R^2, then $codim(X \cap Z) = 2$ and $codim(X) + codim(Z) = 1 + 1 = 2$.

2.(G.P 5.8) For which values of a does the hyperboloid defined by $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = a$ transversally? What does the intersection look like for different values of a?

Solution. Consult Figure 3.

Clearly $a \geq 0$. The hyperboloid intersects the sphere transversally for $0 \leq a < 1$ (vacuously) and $a > 1$.

For $0 \leq a < 1$ the intersection is the empty set.

For $a = 1$ the intersection is the unit circle $x^2 + y^2 = 1$.

For $a > 1$ the intersection is two circles of radius $\sqrt{\frac{1+a}{2}}$ in the $x - y$ plane with centres at $(0, 0, \pm\sqrt{\frac{a-1}{2}})$.

3. (G.P 6.7) Show that the antipodal map $x \to -x$ of $S^k \to S^k$ is homotopic to the identity if k is odd. Hint: Start off with $k = 1$ by using the linear maps defined by

$$\begin{pmatrix} cos\pi t & -sin\pi t \\ sin\pi t & cos\pi t \end{pmatrix}$$

Solution. For odd n, S^n is a submanifold in $\mathbb{R}^{n+1}$ so points on S^n can be described as Euclidean $(n+1)-$tuples, $(x_0, x_1, \dots, x_n)$. Note $n + 1$ is even.

Define $R_i : \mathbb{R}^2 \to \mathbb{R}^2$ as the linear map

$$\begin{pmatrix} x_{2i} \\ x_{2i+1} \end{pmatrix} = \begin{pmatrix} cos(\pi t) & -sin(\pi t) \\ sin(\pi t) & cos(\pi t) \end{pmatrix} \begin{pmatrix} x_{2i} \\ x_{2i+1} \end{pmatrix}$$

for $t \in [0, 1]$. Clearly for $t = 0$ this is equal to the identity map and for $t = 1$ the identity map. Furthermore, these are rotation matrices and thus allowing t to span $[0, 1]$ smoothly transforms R_i, i.e. we can define a homotopy $H_i : \mathbb{R}^2 \times I \to \mathbb{R}^2$ as R_i.

Then define a homotopy on S^n as

$$H = H_0 \times H_1 \times \cdots \times H_{(n+1)/2}$$

i.e. let a rotation operator as defined above act on each pair $(x_i, x_{i+1}$ for even positive intefers i within the $(n + 1)-$tuple. This is a product of smooth functions and thus smooth, and all the conditions above hold for each pair, so this is indeed a homotopy between the identity map and the antipodal map.

4.(G.P 6.10) A deformation of a submanifold Z in Y is smooth homotopy $i_t : Z \to Y$ where i_0 is the inclusion map $Z \to Y$ and each i_t is an embedding. Thus, $Z_t = i_t(Z)$ is a smoothly varying submanifold of Y with $Z_0 = Z$. Show that if Z is compact, then any homotopy i_t of its inclusion map is a deformation for small t. Give an counterexample in the noncompact case (Other than the triviality where $dimZ = dimY$.)

Solution. The inclusion map $i : Z \to Y$ is an embedding. Since the embedding is a stable property, and Z is compact, then any homotopy i_t of its inclusion map is a deformation for small t.

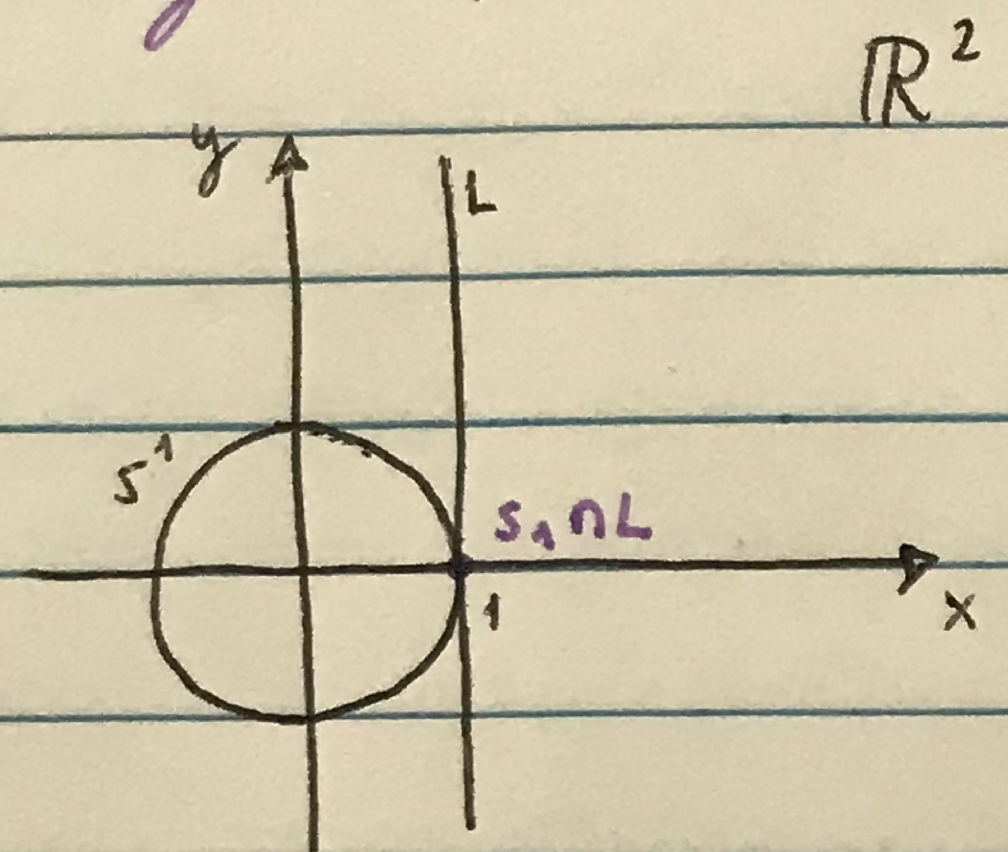
Counterexample: Consider $Z = Im(f)$ where $f : R \to R^2$ is the function we defined in class as:

$$f(x) = \begin{cases} (x, \frac{sin(x)}{x}) & x \neq 0 \\ \text{(0,1)} & \text{x} = 0 \end{cases}$$

It is easy to see this Z is not compact and $dim(Z) = 1 \neq dim(Y)$. And there exists a homotopy i_t such that i_t is not embedding. To see this, since the image of f gets arbitrarily close to the x-axis, with small perturbation, the differential will be 0 when taken at the x-axis. Hence, not an embedding.

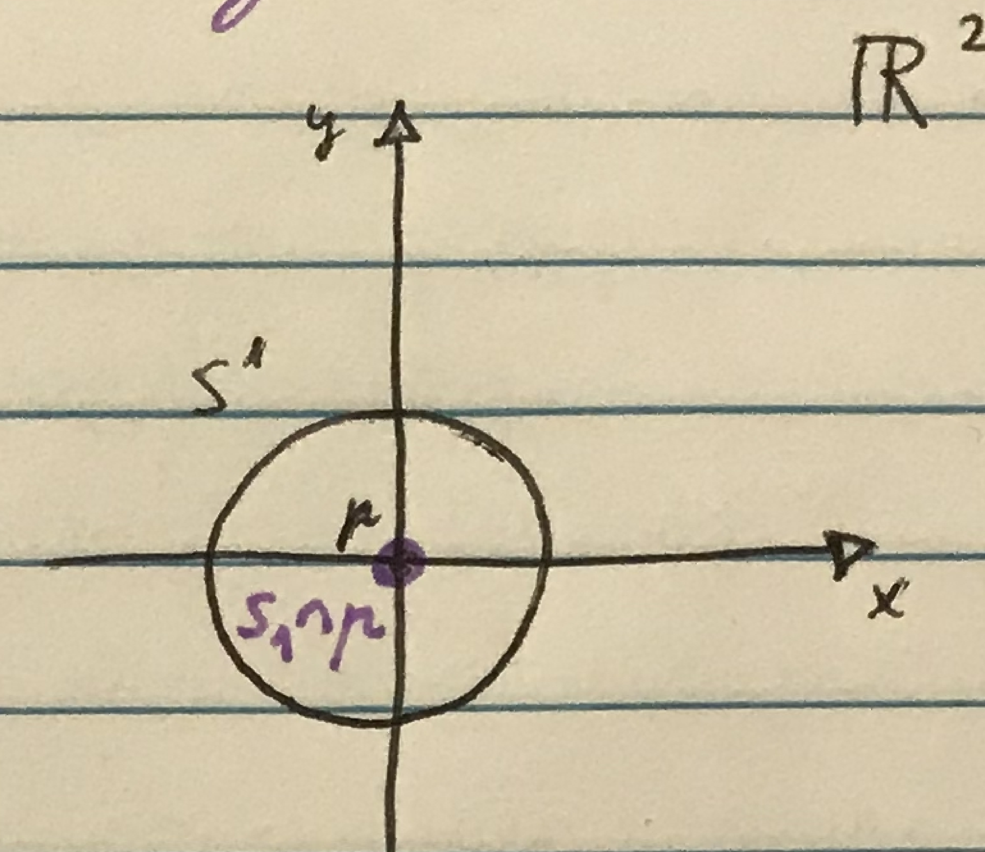
Differential Topology
Homework 7 - Extra Drawings

6 Figure 1



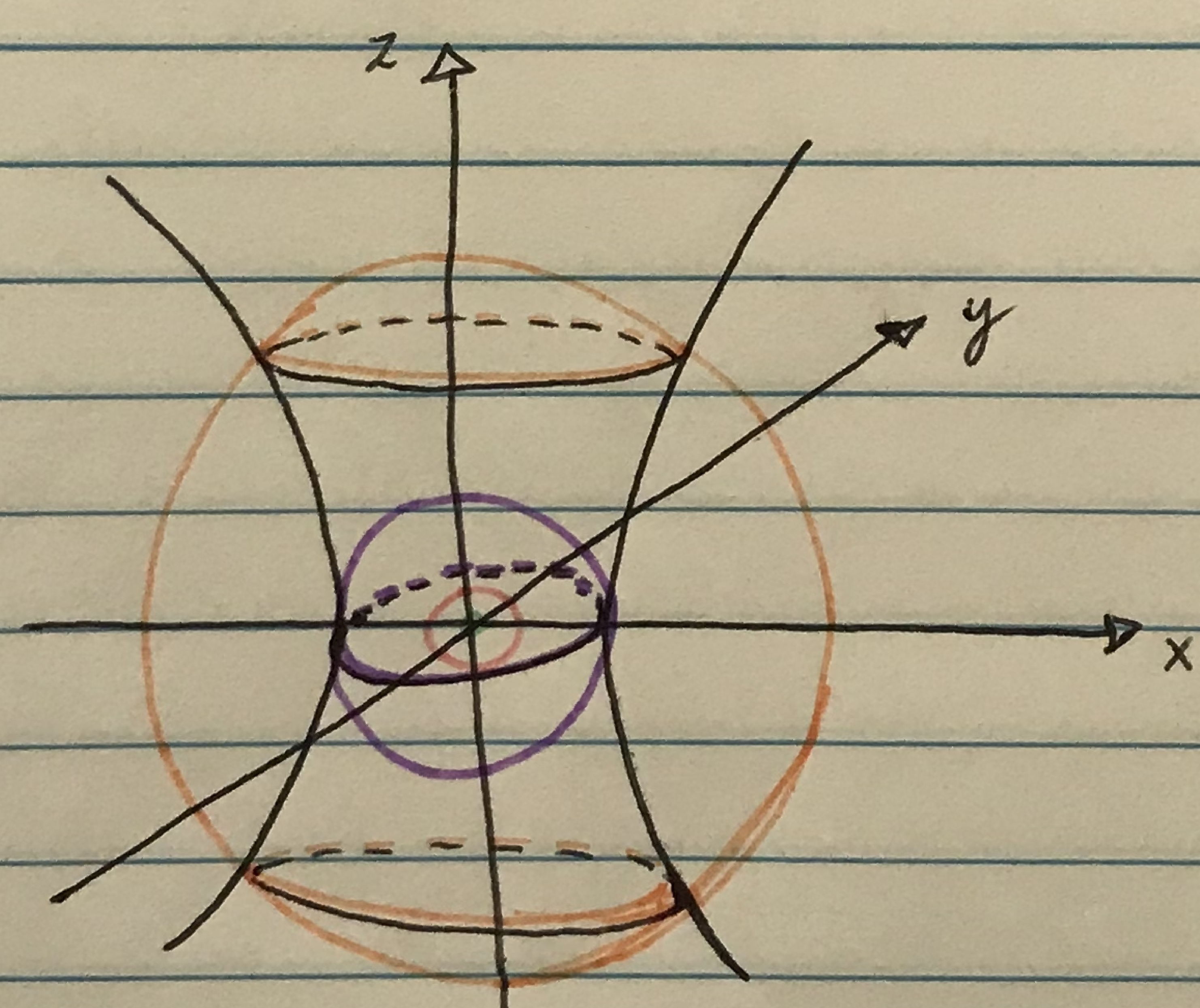
L : line $x = 1$
$S^1, L \subset \mathbb{R}^2$

Figure 2



p: point $(0,0)$
$S^1, L \subset \mathbb{R}^2$

8 Figure 3



	$x^2+y^2+z^2=a$	intersection
$a=0$	point	$\emptyset$
$0<a<1$	sphere	$\emptyset$
$a=1$	sphere	1 circle
$a>1$	sphere	2 circles

$a=1$ $\quad x^2+y^2-z^2=1$
$\& \; x^2+y^2+z^2=1 \implies x^2+y^2=1, \; z=0$

$a>1$ $\quad x^2+y^2-z^2=1$
$\& \; x^2+y^2+z^2=a \implies x^2+y^2=\frac{1+a}{2}, \; z^2=\frac{a-1}{2}$
$\implies z = \pm\sqrt{\frac{a-1}{2}}$