Math 141 Homework 5

1. GP5, 1 and 2

1. (a) By the definition, for an arbitrary point q in $\mathbb{R}^n$, for any point p in the pre-image of q, $T_qV + d_pA(T_p\mathbb{R}^k) = T_q\mathbb{R}^n$, which in this case is just $V + A(\mathbb{R}^k) = \mathbb{R}^n$, because V is a vector subspace and thus diffeomorphic to a Euclidean space. Thus, these two definitions are equivalent.

   (b) The formal definition gives us that $T_qV + d_pi(T_pW) = T_q\mathbb{R}^n$ where $i : W \to \mathbb{R}^n$ the inclusion map which is clearly linear. Knowing that these are linear, we can simplify this down to $V + i(W) = \mathbb{R}^n = V + W$. Thus, these two definitions are equivalent.

2. (a), (b), (d), (e), (f), (g). I applied result from 1(b) to all of these problems implicitly.

   (a) Yes, these clearly span $\mathbb{R}^3$.

   (b) Yes, these also span $\mathbb{R}^3$ since the second plane has z component.

   (c) No, this is the xy-plane and y axis, which do not span in the z direction.

   (d) Yes, if $k + l \geq n$. We see that these spaces must hit every index of n which is only possible when $k + l \geq n$.

   (e) Yes, only if $\max(k, l) = n$. The minimal dimension is covered by the larger, so consider if $\mathbb{R}^k \times \{0\}$ spans $\mathbb{R}^n$ which is only true if $k = n$ and the spaces are equal. This is only possible depending on how we are considering the $\{0\}$ notation (if it must be nontrivial, then this is never transversal).

   (f) Yes, because the first gives it half of the dimension and span, whereas the diagonal contributes the other half. Using linear algebra, you can separate them into (d).

   (g) Yes, the dimensions add up correctly ($n + 2n(n-1)/2 = n^2$), and they are independent (the only intersection is zero-matrix that equals its negative), so they must span $\mathbb{R}^{n^2}$.

2. GP6, 1 and 2. You may use the function ρ in the hint for problem 1 as given.

1. Using ρ, a function we created in a previous homework using fraction of integrals, from $[0, 1/4]$, the smooth function ρ would give us $\rho(t) = 0$, and from $[3/4, 1]$, $\rho(t) = 1$. Now, we can just take F a homotopy, which exists since f_0 and f_1 are given to be homotopic. Considering $\tilde{F}(x, t) = F(x, \rho(t))$, we have that $\tilde{F}(x, t) = f_0(x)$ for $t \in [0, 1/4]$ since the actions are suppressed by $\rho(t) = 0$, and also $\tilde{F}(x, t) = f_1(x)$ for $t \in [3/4, 1]$ since $\rho(t) = 1$. $\tilde{F}$ is still a smooth map since it is the composition of smooth maps (it is $F \circ (Id \times \rho)$).

2. Both reflexivity and symmetry are clear. Transitivity: If $f \sim g$ and $g \sim h$, then there exist homotopies $F : X \times I \to Y$ and $G : X \times I \to Y$. Note that the domain and ranges of these functions must be the same for them to be homotopic, so we have labelled X to be the domain and Y to be the range. Now, define a homotopy H using problem 1 such that $H(x, t) = \tilde{F}(x, t)$ for $t \in [0, 1]$ and $\tilde{G}(x, t-1)$ for $t \in [1, 2]$. Now, to adjust for the larger interval, consider $\tilde{H} = H(x, t/2)$. $\tilde{H}(x, 1/2) = H(x, 1) = F(x, 1) = g = G(x, 0)$ so this is continuous. Additionally, $\tilde{H}(x, 0) = F(x, 0) = f$ and $\tilde{H}(x, 1) = G(x, 1) = h$. Thus, there is a homotopy between f and h so $f \sim h$.