# Math 141 Homework 5

**Problem 1.** GP4, 9. Show that the orthogonal group $O(n)$ is compact.

**Solution**: From real analysis, being compact in Euclidean space is equivalent to being closed and bounded. We have shown in class that $O(n)$ is just the pre-image of the identity for the function that sends $M$ to $MM^T$, and since the identity is closed, $O(n)$ is as well. We also can show that $O(n)$ is bounded. Consider $\|M\|$ for any $M \in O(n)$, by definition it is

$$\sqrt{\frac{1}{n}\sum_{i,j}(M_{ij})^2} = \sqrt{\frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{n}M_{ij}^2} = \sqrt{\frac{1}{n}\sum_{j=1}^{n}1} = \sqrt{\frac{1}{n} * n} = 1$$

because $M$ is orthogonal. Thus, $O(n)$ is bound by the value 1.

GP4, 10. Verify that the tangent space to $O(n)$ at the identity matrix $I$ is the vector space of skew symmetric $n \times n$ matrices – that is, matrices $A$ satisfying $A^t = -A$.

**Solution**: Let $f : M \to MM^T$. From class, we have the following series of calculations:
$(I + \varepsilon M) * (I + \varepsilon M)^T = I + \varepsilon(M + M^T) + O(\varepsilon^2)$, so $df_I(M) = M + M^T$. By the proposition in the book and by the fact we know that $O(n)$ is the preimage of the identity under $f$, we have that the Kernel of this derivative is the tangent space of $O(n)$, and that is exactly matrices such that $M+M^T = 0 \iff M^T = -M$.

**Problem 2.** GP4, 11
(a) The $n\times n$ matrices with determinant $+1$ form a group denoted $SL(n)$. Prove that $SL(n)$ is a submanifold of $M(n)$ and thus is a Lie group.
(b) Check that the tangent space to $SL(n)$ at the identity matrix consists of all matrices with trace equal to zero.

**Solution**:
(a) We will show this by determining that $SL_n$ is the pre-image of the regular value 1 under the determinant function. The differential of the determinant function is a linear map into $\mathbb{R}$, and thus as long as the differential is not the 0 function, it must be surjective. To confirm that the differential of the determinant is never the 0 function, let us consider $d(det)_M(A)$ for an arbitrary matrix $M$. Assume that $det(A) \neq 0$, then $det(M + \varepsilon A) \geq det(M) + \varepsilon det(A) \neq det(M)$ for $\varepsilon > 0$, and thus, the derivative is not 0. We can conclude that 1 is a regular value, and by the Pre-image Theorem, we have that $SL_n$ is indeed a submanifold.

(b) From class, we have that $d_I(det)(M) = Tr(M)$. By the proposition in the section 4, and knowing that $SL(n)$ is the pre-image of 1 under $det$, we have: $T_I(SL(n)) = Ker(d(det)_I) = \{M : Tr(M) = 0\}$.

**Problem 3.** Recall that a Lie group is a *subgroup* of some $GL_n$ which is a manifold. Suppose $f : G \to H$ is a map of Lie groups which is both a smooth map and a *homomorphism*, i.e. such that $f(g_1 \cdot g_2) = f(g_1) \cdot f(g_2)$. Show that $f$ is a (submersion, immersion, local diffeomorphism) if and only if $d_e f$ (the differential at the identity) is, respectively, a (surjective, injective, bijective) linear map. Give an example of a map of connected

Lie groups which is a local diffeomorphism but not a diffeomorphism. (Hint: start by showing that if $G$ is *abelian*, such as $SO(2)$, then the map $g \mapsto g^2$ is a homomorphism. More interesting non-abelian examples also exist.)

**Solution.** Let $f : G \to H$ be a map of Lie groups which is both a smooth map and a homomorphism, i.e. such that $f(g_1 \cdot g_2) = f(g_1) \cdot f(g_2)$, and $f$ is a submersion. Then $d_e f$ is surjective by definition, similarly for immersion and injectivity.

Now suppose $f$ is smooth and a homomorphism, as well as $d_e f$ being surjective. Then we can locally draw the diagram, for neighbourhoods $N_{e_G}$ of $e_G$ in $G$ annd $N_{e_G}$ of $f(e_G) = e_H$ in $H$,

$$\begin{array}{ccc} N_{e_G} & \xrightarrow{f} & N_{e_H} \\ \phi\uparrow & & \psi\uparrow \\ \mathbb{R}^{n^2} & \xrightarrow{d_e f} & \mathbb{R}^{m^2} \end{array}$$

for $n \geq m$ for surjectivity.

Note that left translation $L_x$ in $G$ is smooth, as it is a restriction of the smooth multiplication map$\mu$, $L_x = \mu|_{\{g\}\times G}$. Furthermore, for any $g_1, g_2 \in G$,

$$gg_1 = gg_2 \Rightarrow g^{-1}gg_1 = g^{-1}gg_2 \Rightarrow g_1 = g_2$$

so $L_x$ is injective. Lastly, for any $g_3 \in G$,

$$g_3 = g(g^{-1}g_3)$$

and thus $L_x$ is surjective. Therefore $L_x$ is bijective, and a diffeomorphism.

For any neighbourhood $N_g$ in $G$ of some other point $g \in G$, by left translation by $g^{-1}$ we can map this neighbourhood to a neighbourhood of $e$ under the diffeomorphic left translation. Thus we can construct a diffeomorphism $\phi_2$ between the neighbourhoods $N_g$ and $N_{e_G}$ and similarly, as $H$ is also a Lie group, a diffeomorphism $\psi_2$ between $f(N_g) = N_{e_H}$ by homomorphism, and $N_{e_H}$. Thus the composition $\phi_2^{-1} \circ \phi^{-1} \circ d_e f \circ \psi \circ \psi_2$ is surjective (all diffeomorphisms, $d_e f$ surjective) and thus $f : N_g \to N_{f(g)}$ is surjective, at which point by constructing the standard commutative diagram with diffeomorphic parametrisations from the tangent space, we see that $d_g f$ must also be surjective.

Similarly, replacing the condition of surjectivity by injectivity, we can construct all of these diffeomorphisms (which are injective by definition) and deduce $d_g f$ must be injective at any point, i.e. f an immersion.

Lastly, local diffeomorphisms are locally bijective by definition. For the converse, we have shown surjectivity and injectivity to imply submersion and immersion respectively, hence bijectivity at $e$ implies local injectivity and surjectivity of the derivative at any point, i.e. bijectivity.

Let $sq : SO(2) \to SO(2)$ be the squaring map on the *commutative* (a.k.a. Abelian) group $SO(2)$ of rotations of the plane. Then for $M, N \in SO(2)$ we have $sq(MN) := (MN)^2 = MNMN = M^2N^2 = sq(M)sq(N)$, so this map is a homomorphism. As we have seen previously, this map is a local diffeomorphism at the identity, therefore (since it is a homomorphism), a local diffeomorphism everywhere. On the other hand, it is not a bijection, since

$$sq\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = sq\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = I.$$

**Problem 4.**

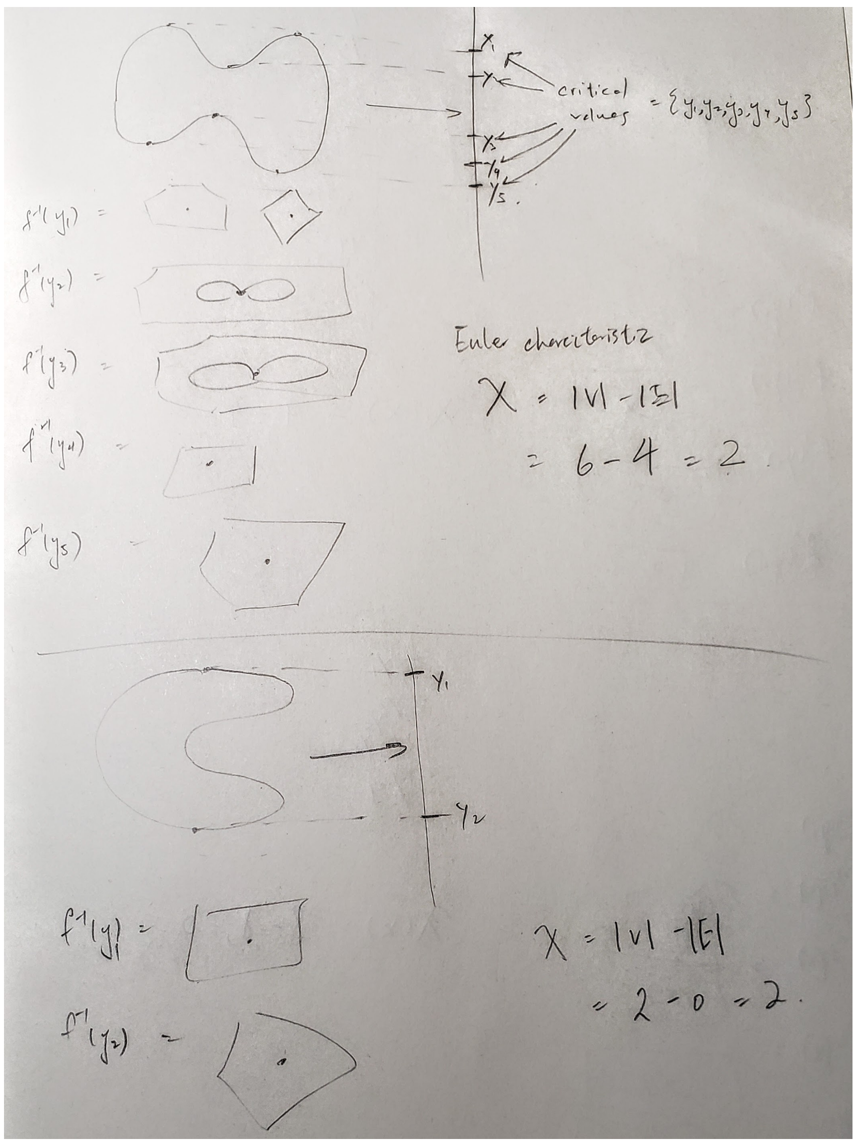
1. Draw or write down two maps $X \to \mathbb{R}$ with different but finite numbers of critical values, for two manifolds $X$ diffeomorphic to the sphere. Draw a picture of the map and the pre-image of each critical value, and compute the Euler characteristic of each resulting one-dimensional manifold (defined as $V - E$). Check that in each case the Euler characteristics sum up to 2, which is the Euler characteristic of $S^2$.

2. Do the same thing with $X$ diffeomorphic to the torus, $S^1 \times S^1$.

   Hint for drawing maps to $\mathbb{R}$ : start by drawing a shape in $\mathbb{R}^3$ which can be smoothly deformed to a sphere or a torus, then use projection to the $z$-axis to define the map.

   **Solution.**

   (a) See picture below. Correction to second computation: should be 4 (from singleton points) $-2$ (from figure eights) $= 2$



(b) See Picture below. Correction to computations: first one should be 2 (from singleton points) $-2$ (from figure 8's) $= 0$ and second one should similarly be $4 - 4 = 0$.