Math 141 Homework 4

October 6, 2019

1. (a) From Exercise 9 in Section 2, we recall

$$d(f \times g)_{(x,y)} = df_x \times dg_y.$$

Therefore if $d(f \times g)_{(x,y)}(a, b) = 0$, then

$$df_x(a) = 0, \quad dg_y(b) = 0.$$

As both f and g are immersions, this implies that $a = 0$ and $b = 0$. In other words, $d(f \times g)_{(x,y)}$ is injective. Hence $f \times g$ is an immersion.

(b) Suppose $d(g \circ f)_x(a) = dg_{f(x)} \circ df_x(a) = 0$. If $a \neq 0$, then $df_x(a) \neq 0$ as f is an immersion. However, by the same reasoning, this implies $dg_{f(x)} \circ df_x(a) \neq 0$. Hence $d(g \circ f)_x(a) = 0$ only if $a = 0$, implying that $d(g \circ f)_x$ is injective. Because our choice of x was arbitrary, we conclude that $g \circ f$ is an immersion.

(c) Suppose $f : X \to Y$ and Z is a submanifold of X. Denote $\iota : Z \to X$ as the inclusion map $z \mapsto z$, and let $g : Z \to Y$ be the restriction of f to Z. By the chain rule, we note

$$dg_z = df_z \circ d\iota_z = df_z(I) = df_z$$

for all $z \in Z$. Hence dg_z is injective, and g is a immersion.

(d) Recall $T_x(X)$ and $T_f(x)(Y)$ have the same dimensions as X and Y. Thus df_x is an injective mapping between vector spaces of equal dimension, i.e. df_x is a isomorphism. From the inverse funcion theorem, we conclude that f is a local diffeomorphism for all $x \in X$.

2. Denote the map as $f = (f_1, f_2)$. To show f is injective, we note $f_1(t) + f_2(t) = e^t$. As e^t is different for different t, this implies $(f_1(t), f_2(t))$ is different for different t.

Now note

$$df_t = \Big(\frac{e^t - e^{-t}}{2}, \frac{e^t + e^{-t}}{2}\Big) = (f_2, f_1).$$

From the previous reasoning, we conclude that df_t is injective, hence f is an immersion.

It remains to show that f is proper. By Heine-Borel, compact subsets of $\mathbb{R}^n$ are precisely those that are closed and bounded. As f is continuous, thus the preimage of closed sets are closed. Hence it suffices to show that the preimages of bounded sets are bounded.

Suppose

$$f_1(t) < a.$$

Note this implies

$$e^t < 2a, \quad e^{-t} < 2a$$

i.e. $|t| < \ln(2a)$, which is a bounded set. As any bounded subset of $\mathbb{R}^2$ is contained in set $(-\infty, a) \times \mathbb{R}$ for some $a \in \mathbb{R}$, thus the preimage via f of any bounded set is bounded.

Having shown that f is an embedding, we observe that

$$f_1(t)^2 - f_2(t)^2 = 1$$

and thus the image of f is contained in the hyperbola $x^2 - y^2 = 1$. Furthermore, by the AM-GM inequality, we conclude

$$f_1(t) \geq 1, \quad \text{with equality when } e^t = e^{-t}.$$

Also, $f_1(t)$ can clearly can arbitrarily large. As $f_1(t)$ is continuous, this implies that $f_1(t)$ can achieve (and only achieve) any value ≥ 1. Finally, we note

$$f(-t) = (f_1(-t), f_2(-t)) = (f_1(t), -f_2(t))$$

so for each $x = f_1(t)$, there are two possible y values (if $y \neq 0$) in the image. From these facts, we conclude that the image of f are precisely the points on the hyperbola $x^2 - y^2 = 1$ with $x \geq 1$, which is one nappe.

3. GP4: 1,2.

   1 Suppose X is a k-dimensional manifold, and Y is an l-dimensional manifold. Let $x \in U$ be arbitrarily picked. It suffices to show that some open neighborhood (contained in U) of x in X maps to an open set in Y.

   Choose a chart $\phi : V \to X$ that parameterizes an open neighborhood of x, and a chart $\psi : W \to Y$ that parameterizes an open neighborhood of $f(x)$. By the Local Submersion Theorem, we can assume without the loss of generality that $\phi(0) = x$, $\psi(0) = f(x)$. and $\psi^{-1} \circ f \circ \phi$ is the canonical submersion π.

   Furthermore, we can assume (without loss of generality) that V is a basis element of the topology of $\mathbb{R}^k$ and $f(V) \subset U$ (both can be done by shrinking V). Then $\pi(V)$ is a basis element of the topology of $\mathbb{R}^l$. As ψ is a diffeomorphism, we conclude that

   $$\psi \circ \pi(V)$$

   is open in Y, i.e.

   $$f(\phi(V))$$

   is open in Y, which is what we set out to prove.

   2 (a) From the previous problem, we note $f(X)$ is open in Y. However, we recall the that continuous images of compact spaces are compact, hence $f(X)$ is compact. As compactness is independent of ambient space, this implies $f(X)$ is compact for $\mathbb{R}^n \supset Y$, i.e. $f(X)$ is closed in $\mathbb{R}^n$. Hence $f(X)$ is also closed in Y. As $f(X)$ is both open and closed, and Y is connected, this implies that $f(X) = Y$ or $f(X) = \emptyset$, the latter of which is impossible.

   (b) As Euclidean spaces are connected, we conclude that any submersion of a compact manifold into an Euclidean space is surjective. However, this implies that the continuous image of a compact manifold is an Euclidean space, which is a contradiction as Euclidean spaces are not compact.

4. GP4: 3, 5.

   3 Note $f : t \mapsto (t, t^2, t^3)$ and

$$df_t = (1, 2t, 3t^2)$$

   are both injective. Note as f is continuous, it takes the preimages of closed sets to closed sets. Furthermore, suppose some set $S \subset f(\mathbb{R})$ is bounded. Then there exists $a < b$ such that

$$S \subset X = ((a, b) \times \mathbb{R} \times \mathbb{R}) \cap f(\mathbb{R}).$$

   Note $f^{-1}(X) = (a, b)$, which is bounded. Thus the preimages of bounded sets are bounded, and we conclude (by Heine-Borel) that f is proper. Hence f is an embedding.

   Define $g : \mathbb{R}^3 \to \mathbb{R}$ and $h : \mathbb{R}^3 \to \mathbb{R}$ given by

$$g(a, b, c) = a^2 - b, \quad g(a, b, c) = ab - c.$$

   Clearly $f(\mathbb{R}) = g^{-1}(0) \cap h^{-1}(0)$. To show that f and g are independent, we note

$$dg_{(a,b,c)} = (2a, -1, 0), \quad dh_{(a,b,c)} = (b, a, -1).$$

   A comparsion of the third coordinate tells us that one will never be a multiple of the other, so g and h are independent on all of $\mathbb{R}^3$.

   5 We observe

$$df_{(x,y,z)} = (2x, 2y, -2z)$$

   which is surjective unless $x = y = z = 0$. Thus every value of $\mathbb{R}$ is a critical value except 0.

   Suppose a and b are of the same sign, i.e. $a/b > 0$. Consider the diffeomorphic smooth map $g : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$g(x, y, z) = (xn, yn, zn), \quad n = \sqrt{\frac{a}{b}}.$$

   Note $g(f^{-1}(b)) \subset f^{-1}(a)$ and $g^{-1}(f^{-1}(a)) \subset f^{-1}(b)$, hence $g(f^{-1}(a)) = f^{-1}(b)$. As diffeomorphisms restricted to smaller domains are still diffeomorphisms, we conclude that $f^{-1}(a)$ is diffeomorphic to $f^{-1}(b)$.

   Pictorical Description. For positive a, the manifold $f^{-1}(a)$ is shaped like an hourglass centered at the origin. As a approaches 0, the neck of the hourglass gets thinner until the set $f^{-1}(0)$ is just two oppositive facing cones whose tips touch at the origin. When a becomes negative, $f^{-1}(a)$ becomes a disconnected manifold (the two cones separate from each other and the tips smooth out).

5. GP4: 12, 13.

   12 Note the determinant function can be rewritten as a mapping between manifolds $f : \mathbb{R}^4 \backslash \{0\} \to \mathbb{R}$ given by

$$f(a, b, c, d) = ac - bd.$$

   Note

$$df_{(a,b,c,d)} = (c, -d, a, -b)$$

is always surjective (as not all $a, b, c, d = 0$), implying that f is a submersion. In particular, this implies that any value is a regular value. Applying the preimage theorem, we conclude that $f^{-1}(0)$ is a submanifold. However, this is precisely the nonzero noninvertible matrices, i.e. the matrices of rank 1. Therefore, the matrices of rank 1 form a submanifold.

13 We first concern ourselves with the union of manifolds.

Lemma 1. The union of manifolds of the same dimension

$$X = \bigcup_{i \in I} X_i$$

is a manifold if for every $x \in X$, there exists an X_i such that a neighborhood N of x in X is a neighborhood of x in X_i.

Proof. Choose a parameterization about x in X_i such that the parameterization maps into N. This chart is also a chart for X. As we can do this for every $x \in X$, thus X is a manifold.

Lemma 2. Every $m \times n$ matrix with rank r has a $r \times r$ nonsingular minor.

Proof. Suppose our matrix is (a_{ij}) where $1 \leq i \leq m$ and $1 \leq j \leq n$. Choose linearly independent rows $i_1, i_2, \ldots i_r$ and linearly independent columns $j_1, j_2, \ldots, j_r$. Then the minor (a_{ij}) where $i \in \{i_1, \ldots, i_r\}$ and $j \in \{j_1, \ldots, j_r\}$ is invertible.

Back to Problem. However, such a minor can be found anywhere in a $m \times n$ matrix of rank r. If $I \subset [1, m]$ and $J \subset [1, n]$ are both sets of r integers, we denote M_{IJ} as all the $m \times n$ matrices such that

$$(a_{ij}), \quad i \in I, \quad j \in J$$

is an invertible minor. Then the union

$$\bigcup_{I,J} M_{I,J}$$

consists of all matrices that have some $r \times r$ nonsingular minor.

Lemma 3. Each M_{IJ} is an open set.

Proof. Define f to be the map that computes the determinant of the minor

$$(a_{ij}), \quad i \in I, \quad j \in J.$$

Then $M_{IJ} = f^{-1}(\mathbb{R} \backslash \{0\})$. As f is a polynomial function of the entries of an $m \times n$ matrix, it is continuous and thus takes the preimages of open sets to open sets.

Back to Problem. Now we take into account the book's hint. Consider the matrices whose upper right $r \times r$ minor is nonsingular, i.e. $M_{[1,r][1,r]}$. We use the book's notation and represent these matrices as

$$\begin{pmatrix} B & C \\ D & E \end{pmatrix}.$$

Then if we post mulitply by a nonsingular matrix

$$\begin{pmatrix} I & -B^{-1}C \\ 0 & I \end{pmatrix}$$

we get the product

$$\begin{pmatrix} B & 0 \\ D & E - DB^{-1}C. \end{pmatrix}$$

If our original matrix has rank r, then so does our final matrix, and $E - DB^{-1}C = 0$. Conversely, suppose $E - DB^{-1}C = 0$. Then our product matrix has rank r, and hence so does the orignal matrix. Therefore, the matrices in $M_{[1,r][1,r]}$ with rank r are precisely those where $E - DB^{-1}C = 0$. We can rewrite

$$E - DB^{-1}C = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1(n-r)} \\ g_{21} & g_{22} & \cdots & g_{2(n-r)} \\ \cdots & \cdots & \cdots & \cdots \\ g_{(m-r)1} & \cdots & \cdots & g_{(m-r)(n-r)} \end{pmatrix} = 0$$

where each g_{ij} is some polynomial function on the entries of matrices in $M_{[1,r][1,r]}$. Note each g_{ij} is independent from the others because each g_{ij} uniquely takes the ij-th element in E to 1 and every other entry in E to zero. Hence the set of matrices of rank r in $M_{[1,r][1,r]}$ (which we denote as $S_{[1,r][1,r]}$) can be cut out by $(m-r)(n-r)$ independent functions. Thus $S_{[1,r][1,r]}$ is a smooth manifold of codimenison $(m-r)(n-r)$.

Loop. By repeating the logic above for each set M_{IJ} in turn (with computational differences depending on where the minor is located), we note that each $S_{IJ} \subset M_{IJ}$ is a manifold of codimension $(m-r)(n-r)$.

Finale. We claim

$$S = \bigcup_{I,J} S_{IJ}$$

is a manifold. For any $x \in S_{IJ}$, we note $S \cap M_{IJ}$ is a open neighborhood of x in S (Lemma 3). However,

$$S \cap M_{IJ} = S_{IJ}.$$

Hence by Lemma 1, we conclude S is a manifold. By Lemma 2 and the definition of S_{IJ}, we deduce that S is the set of all matrices of rank r. As it is the finite union of submanifolds with codimension $(m-r)(n-r)$, it itself has codimension $(m-r)(n-r)$.

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