Homework 3

Note

If for a point x of a manifold X we write "Let $\phi : U \to X$ be a parametrization around x", we are automatically letting U be open containing the origin, and ϕ map the origin to the point x. We write N_x for the precise open subset of the manifold X which ϕ is a parametrization *of*.

GP2 Exercise 8

Let D be the disc of radius $\sqrt{a}$ centered at the origin of $\mathbb{R}^2$. First we claim that $f(y,z) = \sqrt{z^2 - y^2 + a}$ defines a smooth map $f : D \to \mathbb{R}^+$. This is so because taking partial derivatives only ever involves dividing by (fractional powers of) $z^2 - y^2 - a$, which is positive on this domain. So all the partial derivatives exist.

Therefore $\tilde{f} : D \to \text{graph}(f)$ given by $\tilde{f}(y,z) = (\sqrt{z^2 - y^2 + a}, y, z)$ is a diffeomorphism.

Now we argue that within the open ball B in $\mathbb{R}^3$ of radius $\sqrt{a}$ centered at $(\sqrt{a}, 0, 0)$, all points satisfying the given equation are of the form $(\sqrt{z^2 - y^2 + a}, y, z)$ for some $(y,z) \in D$. Indeed any point $(x,y,z) \in \mathbb{R}^3$ satisfying the equation either looks like this or like $(-\sqrt{z^2 - y^2 + a}, y, z)$, and a point within $\sqrt{a}$ of $(\sqrt{a}, 0, 0)$ cannot have negative x coordinate. We must have $(y,z) \in D$ as $y^2 + z^2 \geq a$ implies that (x,y,z) is at least $\sqrt{a}$ away from $(\sqrt{a}, 0, 0)$ no matter what x is, and thus not in this ball.

It follows that the set of points of B satisfying the equation are precisely $B \cap \tilde{f}(D)$. In other words, the equation defines a submanifold of B diffeomorphic via $\tilde{f}^{-1}$ to some open subset of D. We know $\tilde{f}$ maps the origin to $(\sqrt{a}, 0, 0)$, so the tangent space at this point will be $\text{im}(d\tilde{f}_0)$.

We easily calculate $d\tilde{f}_0$ in the standard basis as

$$\begin{bmatrix} \frac{\partial(\sqrt{z^2-y^2+a})}{\partial y} & \frac{\partial(\sqrt{z^2-y^2+a})}{\partial z} \\ \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{-y}{(z^2-y^2+a)^{3/2}} & \frac{z}{(z^2-y^2+a)^{3/2}} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

at $y = 0$, $z = 0$. This has image the yz-plane, which we can see geometrically is indeed the tangent space here.

GP2 Exercise 10

a) Let $\phi : U \to X$ be a parametrization around $x \in X$. It is a fact that $\phi \times \phi$ is a parametrization $: U \times U \to X \times X$ around (x,x). (Injectivity and surjectivity are purely set-theoretic. Smoothness of $\phi \times \phi$ and its inverse, $\phi^{-1} \times \phi^{-1}$, follow from §1 Exercise 14.) We may calculate df_x using these two parametrizations. Letting $\alpha = (\phi^{-1} \times \phi^{-1}) \circ f \circ \phi$, we observe that α is just the diagonal map $: U \to U \times U$.

We have that df_x is equal to $d(\phi^{-1} \times \phi^{-1})_0 \circ d\alpha_0 \circ (d\phi_0)^{-1}$ by definition.

By the above α is actually linear on its domain, so the derivative $d\alpha_x$ at any point x (in particular $x = 0$) is just α itself, extended linearly to all of $\mathbb{R}^k$. But then it is easy to compute that the above composition sends $v \mapsto (v, v)$. $\square$

b) With the same maps as in the previous part, we have that $f \circ \phi$ is a parametrization of the diagonal Δ around (x, x). By the manifold chain rule we have $d(f \circ \phi)_0 = df_x \circ d\phi_0$. We know $d\phi_0$ is an isomorphism to $T_x(X)$; combining this with the calculation of df_x in the previous part (and basic linear algebra) we have $\operatorname{im}(df_x \circ d\phi_0) = T_x(X) \times T_x(X)$. This is $T_{(x,x)}(\Delta)$. $\square$

GP2 Exercise 11

a) Let the diagonal map $: X \to X \times X$ of the last exercise now be called δ. Then using the product-of-maps notation of Exercise 9 part (d), we have $F = (\mathrm{id} \times f) \circ \delta$. Thus by the manifold chain rule $dF_x = d(\mathrm{id} \times f)_{(x,x)} \circ d\delta_x$.

By part (a) of the last exercise, $d\delta_x$ justs sends any $v \in T_x(X)$ to (v, v). By Exercise 9 part (d), $d(\mathrm{id} \times f)_{(x,x)} = d\mathrm{id}_x \times df_x = \mathrm{id} \times df_x$.

Putting both of these together, we see the composition $d(\mathrm{id} \times f)_{(x,x)} \circ d\delta_x$ indeed sends $v \mapsto (v, df_x(v))$. $\square$

b) The diffeomorphic copy of X inside $X \times Y$ given by the image of this diffeomorphism F, is what we define to be $\mathrm{graph}(f)$. The tangent space $T_{F(x)}(\mathrm{graph}(f))$ to it at a point $F(x)$ is then $\operatorname{im}(dF_x)$. By the previous part's calculation of dF_x, this is exactly $\{(v, df_x(v))\}$, which is $\mathrm{graph}(df_x)$. $\square$

GP2 Exercise 12

First check: The curve c is always a map from an open in $\mathbb{R}$ to a subset $X \subset \mathbb{R}^n$. Whether $X = \mathbb{R}^n$ makes no difference; we still must have dc_{t_0} given in the standard basis by the Jacobian, which is $\begin{bmatrix} c_1'(t_0) \\ c_2'(t_0) \\ \vdots \\ c_n'(t_0) \end{bmatrix}$ if $c_i : \mathbb{R} \to \mathbb{R}$ are the coordinate functions. Indeed this returns itself as a vector, $\begin{bmatrix} c_1'(t_0) \\ c_2'(t_0) \\ \vdots \\ c_n'(t_0) \end{bmatrix}$, when applied to the input vector $\begin{bmatrix} 1 \end{bmatrix}$.

Second claim: We first prove the statement for $X = \mathbb{R}^k$. In this case, the tangent space is again $\mathbb{R}^k$, and by the first check any vector

$$\begin{bmatrix} a_1 \\ a_2) \\ \vdots \\ a_n \end{bmatrix}$$

is the velocity vector of the curve $A(x) = (a_1 x, a_2 x, \dots a_n x)$. $\square$

Now if X is any k-dimensional manifold parametrized around x by $\phi : \mathbb{R}^k \to \mathbb{R}^N$, we have an isomorphism of tangent spaces $d\phi_0 : \mathbb{R}^k \to T_x(X)$. To get a curve in X whose velocity vector is $v \in T_x(X)$, take a curve $c(t)$ in $\mathbb{R}^k$ with $c(0) = 0$ whose velocity vector at zero is $(d\phi_0)^{-1}(v)$ (possible by the proof for $X = \mathbb{R}^k$). Then we claim $\phi \circ c$ is a curve in X with velocity vector v at zero. Indeed $d(\phi \circ c)_0(1) = (d\phi_0 \circ dc_0)(1) = d\phi_0(dc_0(1)) = d\phi_0((d\phi_0)^{-1}(v)) = v$. $\square$

GP3 Exercise 7

a) As the map g is smooth, by the inverse function theorem, we need only check that dg is always an isomorphism. But g is a curve $\begin{bmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{bmatrix}$, so by GP 2 Exercise 12 we compute $dg_x = \begin{bmatrix} -2\pi \sin(2\pi x) \\ 2\pi \cos(2\pi x) \end{bmatrix}$ in the standard basis. This is always rank one, so as an operator to $T_x(S^1)$ (which is one-dimensional) it is always an isomorphism. $\square$

b) This is a purely set-theoretic statement, so we identify S^1 with the unit circle in $\mathbb{C}$ for computational convenience. Then G is the map : $\mathbb{R}^2 \to \mathbb{C}^2$ sending $(x, y) \mapsto (e^{2\pi i x}, e^{2\pi i y})$. Suppose G fails to be one-to-one, i.e. we have two distinct points (x, y) and (x', y') mapping under G to the same point. Then

$$e^{2\pi i x} = e^{2\pi i x'}, \qquad e^{2\pi i y} = e^{2\pi i y'}.$$

So

$$e^{2\pi i (x - x')} = e^{2\pi i (y - y')} = 1.$$

But this means $x - x'$ and $y - y'$ are integers. Thus the line from (x, y) to (x', y') has rational slope (or infinite slope). In either case it does not have irrational slope, so we are done. $\square$