MATH 141 Homework 2

1. GP 1.6. $f(x) = x^3$ is smooth, as a polynomial, but f^{-1} is not smooth. To see this, note that $f(y) = \sqrt[3]{y}$, but $(f)'(y) = \frac{1}{3}x^{-\frac{2}{3}}$, so $(f)'(0)$ is not defined, meaning that f is not differentiable at 0, and hence not smooth.

GP 1.7. Let X denote this potential manifold. For X to be a manifold, it must be an n-manifold for some fixed n. We see immediately that $n = 1$ by noting that some points in X have neighbourhoods homeomorphic to 1-dimensional spaces. Then there must be some open set $U \subset X$ containing 0 and an open set $V \subset \mathbb{R}$ that is diffeomorphic to U. Then U is homeomorphic to V. This cannot be true however as removing 0 from U leaves 4 connected components but removing $f(0)$ from V (where f is the homeomorphism from U to V) leaves V with 2 connected components. As homeomorphisms preserve topological properties, including connectedness and connected components, we get a contradiction. Therefore X is not a manifold.

2. GP 1.18.

(a) For $x < 0, f$ is clearly smooth and every derivative is 0 everywhere. For $x > 0$, note that both e^x and $-\frac{1}{x^2}$ are smooth (as x is never 0). Composition of smooth functions is smooth, so we see that f is smooth away from 0. It remains to investigate behaviour at 0. If $f^n(0)$ exists for some positive n, it must be equal to 0. I prove this claim inductively on n, and by taking limits from the left. If $f^{(n)}(0)$ exists, we must have:

$$f^{(n)}(0) = \lim_{x\to 0^-} \frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x} = \lim_{x\to 0^-} \frac{0}{x} = 0.$$

It remains to evaluate these corresponding limits from the right. Differenciating $e^{-x^{-2}}$ iteratively using the product and chain rule, it can be shown that for $x > 0$, $f^{(n)}(x)$ is of the form $p(x)e^{-x^{-2}}$, where $p(x)$ is a polynomial in x . We take the limit (again we inductively assume that $f^{(n-1)}(0) = 0$)

$$\lim_{x\to 0^+} \frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x} = \frac{p(x)e^{-x^{-2}}}{x} = q(x)e^{-x^{-2}}.$$

We now investigate the following limit by the change of variables $x \mapsto z$

$$\lim_{x\to 0^+} x^{-k}e^{-x^{-2}} = \lim_{z\to\infty} z^k e^{-z^2} = \lim_{z\to\infty} \frac{z^k}{e^{z^2}} = \lim_{z\to\infty} \frac{kz^{k-1}}{-2ze^{z^2}} = 0,$$

where we have used L'Hospital's rule, and performed an induction on k. This shows that every term in $q(x)e^{-x^{-2}}$ goes to 0 as $x \to 0^+$, completing the proof of smoothness of f at 0.

(b) There is a typo in GP: instead, define $g(x) = f(x-a)f(b-x)$. Then g is the product of two smooth functions and hence smooth, $g(x) = 0$ if $x \notin (a,b)$ as one of the terms in the product will be 0 as either $x - a \leq 0$ or $b - x \geq 0$, and g will be positive on (a,b) as $x-a, b-x > 0$ and f is positive on $\mathbb{R}_{>0}$.

Note that g is continuous and thus integrable on (a,b) and 0 everywhere else so integrable on $\mathbb{R}$. $\int_{-\infty}^{x} g(t)dt$ has first derivative $g(x)$ at x by the Fundamental Theorem of Calculus and by the smoothness of g. Then every higher order derivative of the integral function must also exist as they are simply the derivatives of g. Finally h must also be smooth as the denominator is a constant.

If $x < a$, $\int_{-\infty}^{x} g(t)dt = \int_{-\infty}^{x} 0dt = 0 \implies h(x) = 0$ for $x < a$. Note that

$$\int_{-\infty}^{\infty} g(t)dt = \int_{-\infty}^{a} g(t)dt + \int_{a}^{b} g(t)dt + \int_{b}^{\infty} g(t)dt = 0 + \int_{a}^{b} g(t)dt + 0,$$

and if $x < b$:

$$\int_{-\infty}^{x} g(t)dt = \int_{a}^{b} g(t)dt + \int_{b}^{x} g(t)dt = \int_{a}^{b} g(t)dt + 0,$$

showing that $h(x) = 1$.

Let $a < x < b$. Then $g(x) = d > 0$ and the continuity of $g \implies \exists \delta > 0$ st $g(z) > \frac{d}{2}$ for $z \in [x-\delta, x]$. This, along with the nonnegativity of g means that $\int_{-\infty}^{x} g(t)dt \geq \int_{x-\delta}^{x} g(t)dt > 0$. So $h(x) > 0$. A very similar argument shows that $h(x)$ is strictly less than 1 for $x < b$. (The fact that h is nondecreasing follows from the nonnegativity of g and shows the non-strict part). So $0 < h(x) < 1$ for $x \in (a,b)$.

(c) Let f, g, h be defined as above but using a^2 and b^2 in place of a and b. Set $i(x) = h(x_1^2 + \cdots + x_k^2)$. Note that $x = (x_1, \ldots, x_k) \mapsto x_1^2 + \cdots x_k^2$ is smooth (as a composition of x^2 and projection functions) so i is smooth. Furthermore $|x| \leq a \implies \sum x_i^2 < a^2 \implies h(\sum x_i^2) = i(x) = 0$. Similarly, $|x| \geq b \implies i(x) = 1$, and by the properties of h, $a < |x| < b \implies 0 < i(x) < 1$. So $1 - i(x)$ is smooth, and is exactly the function we need.

3. GP 2.1. Let $x \in X \subset Y$, and suppose the manifolds are of dimension n in $\mathbb{R}^m$. Then there exists an open nbd of x, $V \subset \mathbb{R}^m$, an open set $U \subset \mathbb{R}^n$ and $f\colon U \to V$ st f is a diffeomorphism, and can be considered as a chart to both X and Y around x. We can then consider the commutative diagram on page 10 on GP, and see that the function $h\colon U \to U$ is the identity map. Taking derivatives, and using the chain rule we see that $df \circ di_x = df \circ dh$, and since df is invertible at all points, $di_x = dh$ which is the identity mapping.

GP 2.2 $T_x(X), T_x(U)$ are independent of the choice of open $V \subset \mathbb{R}^n, W \subset X$, where $x \in W$ and homeomorphism f between them. We can find an open subset of W contained in U, and restrict f to this subset, so we can assume $W \subset U$. Then the same map f maps W to an open nbd of x in both X and U, meaning that $T_x(X) = T_x(U) = \operatorname{im} df_x$.

4. GP 2.6. Let $x = (a, b)$ be given and note that we can let $f : U \to S^1$ be defined by $f(\theta) = (\cos\theta, \sin\theta)$ on some small open interval U contained in $(-\pi, 2\pi)$ in such a way that f is a diffeomorphism from U to some open nbd of (a, b) in S^1. Then $df = (-\sin\theta, \cos\theta)$, and $df_x = (-b, a)$, showing that $T_x(X)$ is the space spanned by $(-b, a)$ as required.

GP 2.7 $T_x(X)$ will be the set of all vectors (x, y, z) perpendicular to (a, b, c) which can be seen to have basis $\{(b, -a, 0), (c, 0, -a)\}$. I am taking "exhibit" to mean we don't have to prove this but the basic idea is to define some parametrisation of S^2 using spherical coordinates, and calculate derivatives as in 2.6.

5. Fix p. Let $\varphi\colon U \to \mathbb{R}^N$ be a diffeomorphism from U to some nbd of p in X st $\varphi(q) = p$. We use the chain rule to show that $f \circ \varphi = 0 \implies df_p \circ \varphi_q = 0$. Therefore for any vector $v \in \operatorname{im}\varphi_q$ we have $df_p(v) = 0$ which means exactly that $T_p(X) = \operatorname{im} d\varphi_q \subset \ker df_p$.

Define $f\colon \mathbb{R}^n \to \mathbb{R}, x = (x_1, \ldots, x_n) \mapsto x_1^2 + \cdots + x_n^2 - 1$. Then f is smooth and $f(X) = 0$. This proves that $T_p(X) \subset \ker df_p$ for all $p \in S^{n-1}$. To show the equality, see that $df_p = 2p^t$. So $df_p(v) = 0 \iff p \cdot v = 0 \iff v \perp p$. For $p \in S^{n-1}$, we have that $T_p(S^{n-1})$ is exactly the set of all vectors v perpendicular to p, proving that $T_p(X) = \ker df_p$.