Math 141 Homework 1

October 6, 2019

1.

2. Hatcher 2, 3.

2(a) Suppose open set U is contained in A. Then for every $x \in U$, A is a neighborhood of x, implying that $x \in \text{int}(A)$. Hence $U \subset \text{int}(A)$ and $\text{int}(A)$ is the largest open set contained by A.

2(b) Suppose A is contained in closed set E and x is a limit point of A. Then x is a limit point of E, and as E is closed, it contains its own limit points. Hence $\bar{A} \subset E$ and $\bar{A}$ is the smallest closed set containing A.

Note. The results of 2(a) and 2(b) will be used frequently in this homework without explicit reference.

3. Note $\emptyset$ and R are in $\mathcal{O}$.

For $I_a, I_b \in \mathcal{O}$, note $I_a \cap I_b = I_{\ \max(a,b)}$. Hence $\mathcal{O}$ is closed under finite intersections. Let $S \subset \mathbb{R} \cup \{-\infty, \infty\}$. Then

$$\bigcup_{s \in S} I_s = I_{\inf(S)}$$

which always exists because $\mathbb{R} \cup \{-\infty, \infty\}$ is a complete totally ordered set ("the least upper bound property"). Hence $\mathcal{O}$ is a topology.

Recall the closure of a set $A \subset \mathbb{R}$ is the smallest closed set containing A. Thus the closure of A under topology $\mathcal{O}$ is $(-\infty, \sup(A)]$.

3. Hatcher 4, 5.

4(a) Suppose $x \in \overline{X - A}$. Then every open neighborhood containing x contains some element of $X - A$. Thus $x \notin \text{int}(A)$, which implies $x \in X - \text{int}(A)$.

Conversely suppose $x \in X - \text{int}(A)$. Then $x \notin \text{int}(A)$, which implies every open set containing x contains some element of $X - A$. Thus $x \in \overline{X - A}$.

4(b) Suppose $x \in \text{int}(X - A)$. Then there exists an open neighborhood of x containing only elements from $X - A$, which implies x in not in the closure of A, i.e. $x \in X - \bar{A}$.

Conversely, if $x \notin \bar{A}$, then there exists a open neighborhood of x in $X - A$, i.e. $x \in \text{int}(X - A)$.

5(a) As $A \cup B \subset \bar{A} \cup \bar{B}$ and $\bar{A} \cup \bar{B}$ is closed, thus $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$ (by exercise 2).

Conversely, the limit points of A are also the limit points of $A \cup B$. Hence $\bar{A} \subset \overline{A \cup B}$ and likewise $\bar{B} \subset \overline{A \cup B}$. Hence $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$.

5(b) As $A \cap B \subset \bar{A} \cap \bar{B}$ and $\bar{A} \cap \bar{B}$ is closed, thus $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$.

As an counterexample for equality, consider the sets $A = [0, 1)$ and $B = (1, 2]$. Then $\overline{A \cap B} = \emptyset$ but $\bar{A} \cap \bar{B} = \{1\}$.

5(c) As $\mathrm{int}(A) \cap \mathrm{int}(B) \subset A \cap B$ and $\mathrm{int}(A) \cap \mathrm{int}(B)$ is open, hence $\mathrm{int}(A) \cap \mathrm{int}(B) \subset \mathrm{int}(A \cap B)$.

Conversely, we note $\mathrm{int}(A \cap B) \subset \mathrm{int}(A)$ and likewise for the roles of A and B switched, hence $\mathrm{int}(A \cap B) \subset \mathrm{int}(A) \cap \mathrm{int}(B)$.

5(d) As $\mathrm{int}(A) \subset \mathrm{int}(A \cup B)$ and $\mathrm{int}(B) \subset \mathrm{int}(A \cup B)$, thus $\mathrm{int}(A) \cup \mathrm{int}(B) \subset \mathrm{int}(A \cup B)$.

As an counterexample for equality, consider the sets $A = (0, 1]$ and $B = [1, 2)$. Then $\mathrm{int}(A) \cup \mathrm{int}(B) = (0, 1) \cup (1, 2)$. However, $\mathrm{int}(A \cup B) = (0, 2)$.

4. Hatcher 7, 8.

   7. Suppose X has a topology $\mathcal{S}$. Then the topology of Y is $\mathcal{T} = \{Y \cap S | S \in \mathcal{S}\}$. As Z is a subspace of Y, thus the topology of Z is $\{Z \cap T | T \in \mathcal{T}\} = \{Z \cap Y \cap S | S \in \mathcal{S}\} = \{Z \cap S | S \in \mathcal{S}\}$, that is, Z has the subspace topology in X.

   8. Recall the subspace topology of a metric space agrees with the topology generated by the metric restricted to that subspace.

      If O is open in the subspace topology, that means for each $x \in O$, there exists an open ball of radius ε centered at x in metric space A such that every point in the ball is contained in O. This is equivalent to saying that there exists an open ball of radius ε centered at x in metric space $\mathbb{R}^2$ such that every point in A is also in O.

      Conversely, suppose for each point $x \in O$, there exists $\varepsilon > 0$ such that every point in A that is ε-close to x is in O. This means the open ball centered at x with radius ε intersected with A is contained in O. By definition of subspace topology, this implies O is a neighborhood of x, hence O is open in A.

5. (a) Consider the family of open disks of radius 1 centered on the lattice points of $\mathbb{R}^2$. As this is an open cover with no finite subcover, $\mathbb{R}^2$ is not compact.

   (b) Recall $\tan(x) : (\frac{-\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is a continuous function. Thus we can define a continuous function $f : \mathbb{R} \times (0, 1) \to \mathbb{R}^2$ by

$$f(a, b) = (a, \tan(\pi b - \frac{\pi}{2})).$$

(Note we have used the property that the composition of continuous functions is continuous). Similarly, we can define a continuous function $g : \mathbb{R}^2 \to \mathbb{R} \times (0, 1)$ by

$$g(a, b) = (a, \frac{\arctan(b)}{\pi} + \frac{1}{2}).$$

It is easy to see that

$$f \circ g = g \circ f = \mathrm{I}$$

hence f is a homeomorphism. From problem 6, we deduce that as $\mathbb{R}^2$ is not compact, then so isn't $\mathbb{R} \times (0, 1)$.

6. Suppose $\{V_i\}_{i\in I}$ is an open cover of $f(X)$. As f is continuous, thus $\{f^{-1}(V_i)\}_{i\in I}$ is an open cover of X. As X is compact, we choose a finite subcover $f^{-1}(V_{a_1}), f^{-1}(V_{a_2}), \cdots, f^{-1}(V_{a_n})$. Then $V_{a_1}, V_{a_2}, \cdots, V_{a_n}$ is a finite subcover of $\{V_i\}_{i\in I}$. Hence $f(X)$ is compact.

7. (a) We denote $\{(a_1, \cdots, a_k, 0, \cdots, 0)\}$ as the set $X \subset \mathbb{R}^l$. Define $\iota : \mathbb{R}^k \to X$ to be the (obvious) diffeomorphism

$$\iota(a_1, \cdots, a_k) = (a_1, \cdots, a_k, 0, \cdots, 0),$$

and $\pi : \mathbb{R}^l \to \mathbb{R}^k$ to be the smooth map

$$\pi((a_1, \cdots, a_l)) = (a_1, \cdots, a_k).$$

Suppose $f = f(a_1, \cdots, a_k)$ is some smooth function defined on $\mathbb{R}^n$. Define g on $\mathbb{R}^l$ by

$$g(x) = f \circ \pi(x).$$

which is smooth as it is the composition of smooth maps on open sets. Thus the restriction of g onto X is smooth. Similarly, for any smooth function g on X, we have the corresponding function

$$f(y) = g \circ \iota(y).$$

As there exists smooth function $\tilde{g}$ defined in a open neighborhood of $\iota(y)$ that agrees with g, we again note that the composition of smooth maps on open sets are smooth and deduce that f is locally smooth, and hence smooth. Therefore the smooth functions on $\mathbb{R}^k$ considered as a subset of $\mathbb{R}^l$ are the "same" as usual.

(b) Let f be a smooth map on X. For any $z \in Z$, as $z \in X$, there exists open set U in $\mathbb{R}^N$ and smooth map $\tilde{f}$ defined on U that agrees with f on $X \cap U$. Thus $\tilde{f}$ agrees with the restriction of f to Z on the set $Z \cap U$, implying that $f|_Z$ is a smooth map on Z.

(c) Given any $x \in X$, let V be an open neighborhood of $f(x)$ in $\mathbb{R}^M$ such that there exists smooth map $\tilde{g}$ defined on V which agrees with g on $Y \cap V$. By definition, there are open neighborhoods U of x such that there exists smooth map $\tilde{f}$ defined on U which agrees with f in $U \cap X$. As f is continuous, we can make U small enough such that range of $\tilde{f}$ is contained in V. Note $\tilde{g} \circ \tilde{f}$ agrees with $g \circ f$ in $U \cap X$ and is smooth. Hence $g \circ f$ is smooth.
If both f and g are diffeomorphisms, we note $(f \circ g)^{-1}$ exists and is equal to $g^{-1} \circ f^{-1}$. However, this is the composition of smooth functions, and (by above) hence smooth. Thus $f \circ g$ is also a diffeomorphism.

8. (a) Via geometric arguments involving similar triangles, we note

$$\pi(a, b, c) = \Big(\frac{a}{1-c}, \frac{b}{1-c}\Big).$$

We show π is smooth by extending it to $\tilde{\pi} : X \to \mathbb{R}^2$ where X is the open set $\{(a, b, c) \in \mathbb{R}^3 | -1 < c < 1\}$ and

$$\tilde{\pi}(a, b, c) = \Big(\frac{a}{1-c}, \frac{b}{1-c}\Big).$$

As $\tilde{\pi}$ is clearly the composition of smooth maps defined on open sets, it is smooth, and hence so is π.

To derive the inverse of π, observe

$$\left(\frac{a}{1-c}\right)^2 + \left(\frac{b}{1-c}\right)^2 = \frac{1-c^2}{(1-c)^2} = \frac{1+c}{1-c}.$$

In other words, if $\pi(a,b,c) = (x,y)$, then

$$x^2 + y^2 = \frac{1+c}{1-c} \implies c = \frac{x^2+y^2-1}{x^2+y^2+1}.$$

Hence

$$\pi^{-1}(x,y) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right)$$

which is once again the composition of smooth maps defined on open sets. Therefore π is a diffeomorphism.

(b) Suppose $p = (x_1, x_2, \cdots, x_{k+1}) \in S^k$. Then we define the stereographic projection $\pi_k : S^k \backslash \{N\} \to \mathbb{R}^k$ as

$$\pi_k(p) = \left(\frac{x_1}{1-x_{k+1}}, \frac{x_2}{1-x_{k+1}}, \cdots, \frac{x_k}{1-x_{k+1}}\right).$$

9.

10. (a) **Usual topology.** As the neighborhood of every rational number contains an irrational number, thus $\bar{S} = \mathbb{R}$.

Trivial topology. The smallest closed set that is nonempty is $\mathbb{R}$, hence the closure is $\mathbb{R}$.

Discrete Topology. Note in the discrete topology, every set is closed, hence $\bar{S} = S$.

(b) Consider the *cocountable* topology, which has opens which are complements of finite or countable sets in $\mathbb{R}$ (together with the empty set). Take S as above. Then as S is uncountable but all closed sets other than $\mathbb{R} = \mathbb{R} \setminus \emptyset$ itself are countable, the only closed set containing S is $\mathbb{R}$, so $\bar{S} = \mathbb{R}$. On the other hand, the closure of any countable set C is C itself, so (as the limit of a sequence is an element of its closure), no sequence of elements of S can have a limit point outside of S.