Math 141 Homework #10 and #11

Vaintrob, 12/3/19

1 Homework 10

1, GP 2.4.1**.** Prove there exists a complex number z such that $f(z) := z^7 + \cos(|z^2|)(1 + 93z^4)$ is equal to 0.
Solution: Write $D^2_N := \{z \mid |z| \leq N\} \subset \mathbb{C}$ for the closed disk of radius N and S^2_N for its boundary, $S^2_N = \partial X$, the circle of radius N. By looking at the asymptotics, it is clear that for large enough real N, if $|z| \geq N$ then

$$|z^7| > \cos(|z^2|)(1 + 93z^4).$$

This implies that, defining $f_t(z) := z^7 + t\cos(|z^2|)(1 + 93z^4)$ gives a homotopy between $f(z)$ and z^7 which never equals to zero on $S^2_N = \partial D^2_N$. From the book, the mod 2 winding number of z^7, when viewed as a function from any circle in $\mathbb{C}$ centered at zero, is 1. As f_t is a homotopy between $f \mid S^2_N$ and $z^7 \mid S^2_N$ as maps $S^2_N \to \mathbb{C} \setminus \{0\}$, their winding numbers (which are intersection numbers in $\mathbb{C} \setminus 0$) are equal, so $W_2(f) \mid S^2_N = 1$. This implies that $f \mid D^2_N$ has 0 in its image. (To see this, you can use the proposition on page 81).

2. GP 2.4.3. Suppose that X and Z are compact manifolds and that $f : X \to Y$, and $g : Z \to Y$ are smooth maps into the manifold Y. If $dimX + dimZ = dimY$, we can define the mod 2 transection number of f and g by $I_2(f, g)$=$I_2(f \times g, \Delta)$ where Δ is the diagonal of $Y \times Y$.

(a) Prove that $I_2(f, g)$ is unaltered if either f or g is varied by a homotopy.

Solution. Suppose f' and g' are homotopic to f and g respectively. Then, from an exercise in chapter 1.6 $f \times g$ is homotopic to $f' \times g'$. Since intersection number is invariant under homotopy, we have that $I_2(f \times g, \Delta) = I_2(f' \times g' \Delta)$. Therefore, $I_2(f, g)$ is unaltered if either f or g is varied by a homotopy.

(b) Check that $I_2(f, g) = I_2(g, f)$.

solution Let $Y' = Y \times Y$. By the hint, define $s : Y' \to Y'$ as $s : (x, y) \to (y, x)$. Then considered the sequence of maps between manifolds $Y' \xrightarrow{f \times g} Y' \xrightarrow{s} Y'$. By the result of exercise 2, we know that

$$I_2(f \times g, s^{-1}(\Delta)) = I_2(s \circ (f \times g), \Delta)$$

Since $s^{-1}(\Delta) = \Delta$ and $s \circ (f \times g) = g \times f$, we have that $I_2(f \times g, \Delta) = I_2(g \times f, \Delta)$.

(c) If Z is actually a submanifold of Y and $i : Z \to Y$ is its inclusion, show that

$$I_2(f, i) = I_2(f, Z)$$

Solution Since i is the inclusion map, we have that $Im(i) = Z$. Therefore, $I_2(f, i) = \#f^{-1}(f(X) \cap i(Z)) = f^{-1}(f(X) \cap Z) = I_2(f, Z)$.

(d)
Solution
If $X \pitchfork Z$, then the result is trivially true.

Assume X is not transversal to Z. From part (c), we know that $I_2(Z,X) = I_2(i_z,X)$ where $i_z : Z \to Y$ is the inclusion map from Z to Y and $I_2(Z,X) = I_2(Z,i_x)$ where $i_x : X \to Y$ is the inclusion map from X to Y. Then, there exists $i'_z : Z \to Y$ homotopic to i_z with the additional property that $i'_z \pitchfork Z$. Similarly, there exists $i'_x : X \to Y$ homotopic to i_x with the additional property that $i'_x \pitchfork Z$. Hence, $I_2(i'_z, i'_x)$ is defined.

From part(a) we know that $I_2(i_z, i_x) = I_2(i'_z, i'_x)$. From part (b), we have $I_2(i'_z, i'_x) = I_2(i'_x, i'_z) = I_2(X,Z)$. Finally, combining all the parts together, we have thus shown that $I_2(X,Z) = I_2(Z,X)$.

3, GP 2.4.5**.** Prove that intersection theory is vacuous in contractable manifolds: if Y is contractable and $\dim Y > 0$, then $I_2(f,Z) = 0$ for every $f : X \to Y, X$ compact and Z closed, $\dim X + \dim Z = \dim Y$. (No dimension-zero anomalies here). In particular, intersection theory is vacuous in Euclidean space.

Solution: Assume $f : X \to Y$ where X is compact and Y is contractable, and we have some $Z \subset Y$ and X and Z had dimensions adding up to Y. Let $I_Y : Y \to Y$, which by definition of contractable, we know is homotopic to some c a constant function. Then, $I_2(f,Z) = I_2(I_Y \circ f, Z) = I_2(c \circ f, Z) = I_2(c,Z) = I_2(c \circ g, Z) = I_2(I_Y \circ g, Z) = I_2(g,Z)$ for any $g : X \to Y$. Further, these values all are 0 since we can always deform the constant function so that it does not intersect Z by choosing some point of Y not in Z. Note that Euclidean space is contractable, and thus this is also true in Euclidean space.

4, GP 2.4.6. Prove that no compact manifold – other than the one-point space – is contractable.

Solution: Assume there is some M that is a compact manifold that is not the one-point space but is contractable. Let $i : M \to Y$ be the inclusion map, and $Z = \{p\}$ for some point $p \in Y$. Then, by exercise 5, $I_2(i,Z) = 0$, however that is not possible because $i(Y) \cap Z = Y \cap Z = Z \neq \emptyset$. The intersection is exactly one point so it is 1 mod 2. Thus, we have a contradiction.

2 Homework 11

1. (G.P 2.4.9) Suppose $f : X \to S^k$ is smooth, where X is compact and $0 < dimX < k$. Then for all closed $Z \subset S^k$ of dimension complementary to X, $I_2(X,Z) = 0$. [HINT: By Sard, there exists $p \notin f(X) \cap Z$. Use stereographic projection, plus Exercise 5]

Solution. Suppose $y \in f(X) \cap Z$ and $f(x) = y$. Since $0 < dimX < k$, we have that $dim(df_x(X)) < k = dim(T_y(S^k))$, i.e $df_x(X) \neq T_y(S^k)$. Therefore, all points in $f(X) \cap Z$ are critical values. By Sard, there has to exist $p \notin f(X) \cap Z$.

By the hint, consider the stereographic projection $P : S^k \to R^k$ from a point $p \in S^k$ and $p \notin f(X) \cap Z$. Let $g : X \to R^k = P \circ f$. For each point x in $f(X) \cap Z$, we can project it to $g(x)$ in R^k. Since P is injective and smooth, $P(Z)$ is a closed submanifold of R^k of the same dimension as Z. Because R^k is contractable, X compact, Z closed with complementary dimension, we have that $I_2(g, P(Z)) = 0 \ mod \ 2$ by Exercise 5. Since $g^{-1}(P(Z)) = f^{-1}(Z)$, we conclude that $I_2(f,Z) = I_2(g, P(Z)) = 0 \ mod \ 2$.

2. (G.P 2.4.10) Prove that S^2 and the torus are not diffeomorphic.

Solution. Without loss of generality, suppose S^2 and the torus are both centered at $(0,0,0)$ and the torus is xy-plane aligned (see the picture below). Then, consider the circles of radii 1, P_1 centered at $(1,0,0)$, P_2 centered at $(-1,0,0)$. Then $I_2(P_1,P_2) = 1 \ mod \ 2$ since the they intersect only at $(0,0,0)$. On the other hand, let C_1, C_2 be two curves in S^2. Let $f : C_1 \to S^2$ be the inclusion map. Since C_1 compact, C_2 closed and $dimC_1 + dimC_2 = 1 + 1 = 2$, by Exercise 9, we have that $I_2(C_1,C_2) = 0 \ mod \ 2$. As a result, S^2 and the torus are not diffeomorphic.