Math 141 Midterm 2 November 24, 2019

Additional sheets available (write your name on any additional sheets!!)

Name: ______________________________

1. Briefly define the following terms.
   (a) $f : X \to Y$ is transversal to $Z \subset Y$.

   (b) A homotopy between smooth maps $f, g : X \to Y$.

   (c) A property $P$ of smooth map $X \to Y$ is stable (or alternatively describe the meaning of stability in the following example: the property of a smooth map $f : X \to Y$ being proper is stable for $X$ compact).

   (d) An $n$-dimensional manifold with boundary (any of our definitions will do).

Extra space

2. (a) Let $\rho \in \mathbb{R} \setminus 0$ be a nonzero real number (a "slope"). Define a map $f_\rho : \mathbb{R} \to S^1 \times S^1$ by

$$f_\rho(t) := (\alpha(t), \alpha(\rho \cdot t))$$

for $\alpha : \mathbb{R} \to S^1$ the standard winding map, $t \mapsto (\cos(t), \sin(t))$. Show that $f_\rho$ is transversal to the circle $Z := S^1 \times \{\theta = 0\}$ (i.e. $Z = \{(x, y, 0, 1) \mid (x, y) \in S^1\}$.

(b) Let $Z' \subset S^1 \times S^1$ be the manifold of pairs of angles $(\theta, \theta')$ satisfying $2\theta \equiv 3\theta' \mod 2\pi$. Show that this is a manifold (hint: consider the map $\tau : (\theta, \theta') \mapsto 2\theta - 3\theta' \mod 2\pi$ from $S^1 \times S^1 \to S^1$. You may assume the map defined in this way is smooth: this can be seen for example by re-writing $\tau$ as $(z_1, z_2) \mapsto z_1^2 \cdot z_2^{-3}$ where $z_1, z_2 \in S^1$ are viewed as unit complex numbers.)

(c) Show that $Z$ and $Z'$ as above are transversal manifolds (hint: there are separate solutions based on either (a) or (b) — be careful in each case to check the relevant transversality/regularity statement).

Extra space

3. The main theorem of mod 2 intersection theory (which we have not proven yet) states that if $f, f' : X \to Y$ are smooth maps of manifolds without boundary with $X$ compact and $f, f'$ are both transversal to $Z \subset Y$ (for $Z \subset Y$ a submanifold without boundary), and $\dim(X) + \dim(Z) = \dim(Y)$, then the number of preimage points $f^{-1}(Z)$ is finite and has the same parity as $(f')^{-1}(Z)$.

   Show that this theorem is not true if $Z$ is assumed to be a *manifold with boundary* (but $X, Y$ do not have boundary). Use the inclusion of $S^1$ in $\mathbb{R}^2$ for the map $f : X \to Y$ to construct your counterexample.

Extra space

4. Quick proofs

(a) Let $X \subset \mathbb{R}^3$ be a one-dimensional manifold. Prove that there exists a point $p \in \mathbb{R}^3$ such that the unit sphere around $p$ intersects $X$ in finitely many points.

(b) Show directly (without using any results from the book or from class) that if $U \subset Y$ is open and $X = \mathrm{pt}$ is a single point then the property of a map $f : X \to Y$ to have image in $U$ is stable.

Extra space

5. True/false (and a one-sentence explanation).

   (a) The intersection of two transversal manifolds is a manifold.

   (b) There is a smooth map $f : \mathbb{R}^2 \to \mathbb{R}^3$ which is "almost surjective", i.e. such that the complement $\mathbb{R}^3 \setminus f(\mathbb{R}^2)$ has measure 0.

   (c) There is a smooth map $f : \mathbb{R} \to \mathbb{R}^3$ which is transversal to the line $\{1, 1, x \mid x \in \mathbb{R}\} \subset \mathbb{R}^3$.

   (d) If $f : X \to Y$ is a submersion then $f$ is transversal to $Z$ for any manifold $Z \subset Y$.

   (e) A measure zero subset of $S^1$ consists of finitely many points.

Extra space