Math 141 | Midterm 2 | November 24, 2019

Additional sheets available (write your name on any additional sheets!!)

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1. Briefly define the following terms.
   (a) $f : X \to Y$ is transversal to $Z \subset Y$.

$\forall \ p \in f^{-1}(Z), \quad T_{f(p)}Z + df(T_pX) = T_pY.$

   (b) A homotopy between smooth maps $f, g : X \to Y$.

$H : X \times I \to Y$ such that

$H(p, 0) = f(p), \quad H(p, 1) = g(p) \ \forall p \in X.$

[we sometimes write $H(-, t)$ as H_t or f_t]

   (c) A property P of smooth map $X \to Y$ is stable (or alternatively describe the meaning of stability in the following example: the property of a smooth map $f : X \to Y$ being proper is stable for X compact).

If f_0 satisfies P and

f_t $(t \in [0, 1])$ is a homotopy with $f_0 = f$

then $\exists \varepsilon > 0$ such that f_t satisfies P $\forall t < \varepsilon$.

   (d) An n-dimensional manifold with boundary (any of our definitions will do).

$X \subset \mathbb{R}^N$ with the property that

each $p \in X$ has neighborhood diffeo to

open subset in $H^n \subset \mathbb{R}^n$

$\uparrow$

$\{(x_1, x_2, \ldots, x_n) \mid x_n \geq 0\}$.

2. (a) Let $\rho \in \mathbb{R} \setminus 0$ be a nonzero real number (a "slope"). Define a map $f_\rho : \mathbb{R} \to S^1 \times S^1$ by

$$f_\rho(t) := (\alpha(t), \alpha(\rho \cdot t))$$

for $\alpha : \mathbb{R} \to S^1$ the standard winding map, $t \mapsto (\cos(t), \sin(t))$. Show that f_ρ is transversal to the circle $Z := S^1 \times \{\theta = 0\}$ (i.e. $Z = \{(x, y, 0, 1) \mid (x, y) \in S^1\}$.

Option 1 Since $\alpha : \mathbb{R} \to S^1$ is a surjective local diffeo, any point $(\alpha(s), \alpha(t)) \in S^1 \times S^1$ has nbd $N_{p,q}$ s.t. $(\alpha, \alpha) : (s-\varepsilon, s+\varepsilon) \times (t-\varepsilon, t+\varepsilon) \to N$ diffeo. Let $t_0 \in f_\rho^{-1}(Z)$. Let $q = f_\rho(t_0) = (\alpha(t_0), \alpha(\rho t_0))$. In $\mathbb{R} \times \mathbb{R}$ the lines $(t, \rho t)$ ← slope ρ and $(t, \rho t_0)$ ← slope 0 intersect transversally => true after applying α to each coordinate.

Option 2: $T_{(x,y,0,1)} Z = \left\langle \begin{pmatrix} -y \\ x \\ 0 \\ 0 \end{pmatrix} \right\rangle$

$f(T_{(x,y)} S^1) = \left\langle \begin{pmatrix} -y \\ x \\ -\rho y \\ \rho x \end{pmatrix} \right\rangle$

Two vectors above are lin. indep for $x^2+y^2=1$ => span is 2-dim in 2-dim space $T_{(x,y,0,1)} S^1 \times S^1$.

(b) Let $Z' \subset S^1 \times S^1$ be the manifold of pairs of angles (θ, θ') satisfying $2\theta \equiv 3\theta' \mod 2\pi$. Show that this is a manifold (hint: consider the map $\tau : (\theta, \theta') \mapsto 2\theta - 3\theta' \mod 2\pi$ from $S^1 \times S^1 \to S^1$. You may assume the map defined in this way is smooth: this can be seen for example by re-writing τ as $(z_1, z_2) \mapsto z_1^2 \cdot z_2^{-3}$ where $z_1, z_2 \in S^1$ are viewed as unit complex numbers.)

Enough to show submersion. Consider the following diagram:

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R} & \longrightarrow & S^1 \times S^1 \\ (x,y) \downarrow \tau & & (\theta,\theta') \downarrow \tau \\ 2x-3y & & 2\theta - 3\theta' \\ \mathbb{R} & \longrightarrow & S^1 \end{array}$$

It is commutative & as left vertical is a linear submersion & horizontal arrows are loc. diffeo, right vertical must be a subm.

(c) Show that Z and Z' as above are transversal manifolds (hint: there are separate solutions based on either (a) or (b) — be careful in each case to check the relevant transversality/regularity statement).

Option 1) $f_{2/3}$ is an immersion (from part (a)) and $\operatorname{Im} f_{2/3} = Z'$. So $f_{2/3} : \mathbb{R} \to Z'$ is a map of 1-dim manifolds with nonzero derivative => a local diffeo. So each $q \in Z'$ has nbd with chart given by $f_{2/3}$ => transversality from part (a) implies $Z \pitchfork Z'$. Cor: $df_{2/3}(t) \subset T_q Z'$ for $t \in f_{2/3}^{-1}(q)$, $\forall q \in Z \cap Z'$. So $T_q Z' + T_q Z \supset df_{2/3}(t) + T_q Z = T_q Y$ by

Option 2 Enough to show $\tau|_Z$ has $\theta = 0$ as regular value. In coords as above:

$$\begin{array}{ccc} \mathbb{R} \times (\alpha, 0,1) & \xrightarrow{\alpha} & Z \\ \downarrow \tau\ 2x & & \downarrow \tau|_Z \\ \mathbb{R} & \longrightarrow & S^1 \end{array}$$

commutative & LHS submersion => RHS submersion.

3. The main theorem of mod 2 intersection theory (which we have not proven yet) states that if $f, f' : X \to Y$ are smooth maps of manifolds without boundary with X compact and f, f' are both transversal to close $\partial \to$ $Z \subset Y$ (for $Z \subset Y$ a submanifold without boundary), and $\dim(X)+\dim(Z) = \dim(Y)$, then the number of preimage points $f^{-1}(Z)$ is finite and has the same parity as $(f')^{-1}(Z)$.

Show that this theorem is not true if Z is assumed to be a *manifold with boundary* (but X, Y do not have boundary). Use the inclusion of S^1 in $\mathbb{R}^2$ for the map $f : X \to Y$ to construct your counterexample.

Let $f = i : S^1 \to \mathbb{R}$ (inclusion)

$f_t = (1+t)\cdot i$

$Z = [-1.5, 5] \times \{0\}$

(closed one-dim manifold with Boundary)

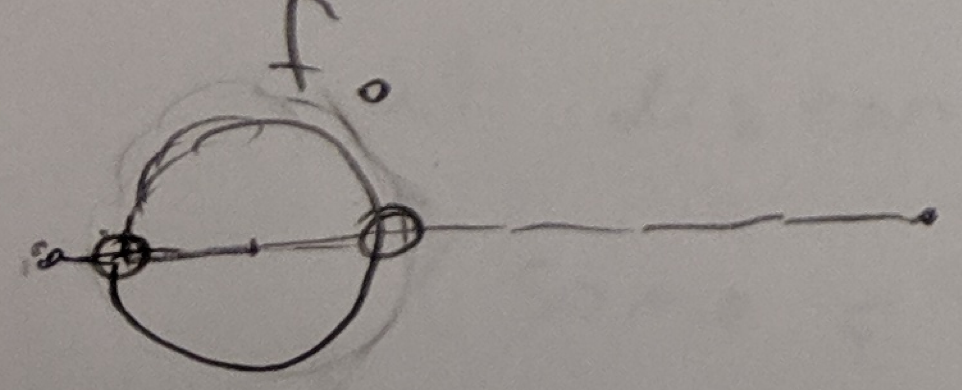
$f_1^{-1}(Z) = \{(-1,0), (1,0)\}$ and

~ 1 pt

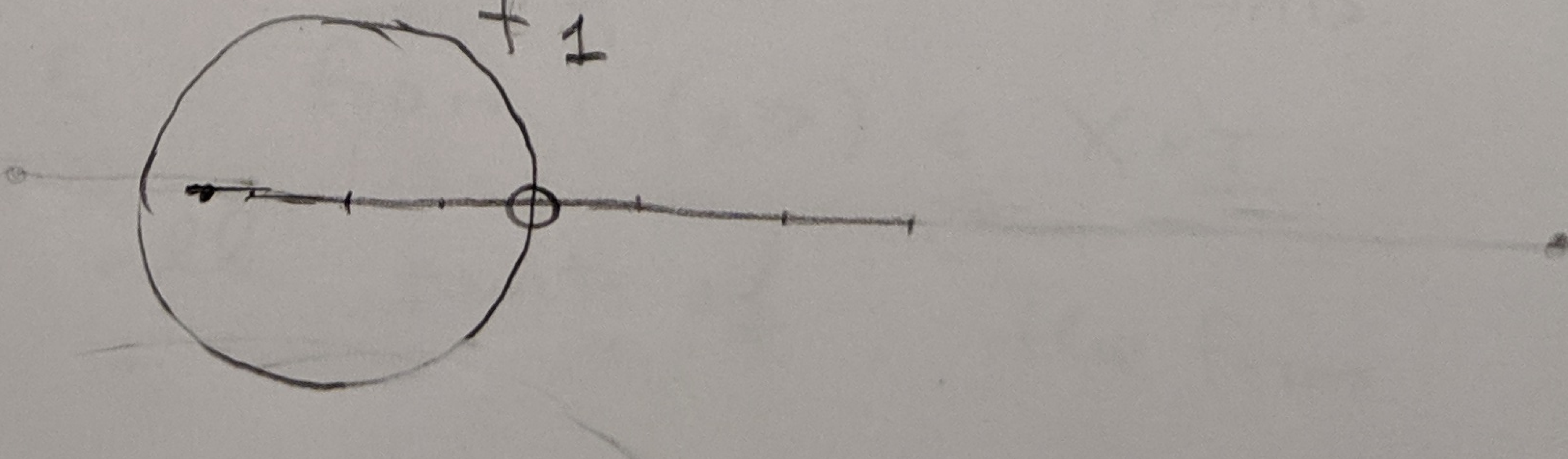
$f_2^{-1}(z) = \{(2,0)\}$ - 2 pts.

(Transversality is clear in both cases ($df(T_p S^1)$ is vertical and $T_q Z$ horizontal at inters. pts):

f_0



f_1



4. Quick proofs

(a) Let $X \subset \mathbb{R}^3$ be a closed one-dimensional manifold. Prove that there exists a point $p \in \mathbb{R}^3$ such that the unit sphere around p intersects X in finitely many points.

Let $f = i_{S^2} : S^2 \to \mathbb{R}^3$

inclusion of unit sphere.

Define $f_p : S^2 \to \mathbb{R}^2$ translated!

$q \mapsto q + p$

By translation transversality, $\exists\, p \in \mathbb{R}^3$

with $f_p \pitchfork X$. $f_p^{-1}(X)$ is a $3-2-1$-

dim compact manifold $\Rightarrow$ finite set.

but $\#\{f_p^{-1}(X)\} = \#(S^2+p) \cap X$ for S^2+p the unit sphere around

(b) Show directly (without using any results from the book or from class) that if $U \subset Y$ is open and $X = \text{pt}$ is a single point then the property of a map $f : X \to Y$ to have image in U is stable.

Let $X = \{p\}$.

A homotopy is a map

$H : X \times I \longrightarrow Y$. Assume $H(p, 0) \in U$.

Then as $H^{-1}(U)$ is open and contains

$(p, 0)$ it must contain all points

at distance $< \varepsilon$ from $(p, 0) \in X \times I$

(some ε) $\Rightarrow$ all points of the form

(p, ε).

5. True/false (and a one-sentence explanation).

(a) The intersection of two transversal manifolds is a manifold.

True. Apply preimage theorem to inclusion map of one of the manifolds.

(b) There is a smooth map $f : \mathbb{R}^2 \to \mathbb{R}^3$ which is "almost surjective", i.e. such that the complement $\mathbb{R}^3 \setminus f(\mathbb{R}^2)$ has measure 0.

False. By dimension deficiency, if $q \in \mathrm{Im} f$ it is a critical value $\Rightarrow$ $\mathrm{Im} f$ has measure 0 & its complement cannot have meas. 0

(c) There is a smooth map $f : \mathbb{R} \to \mathbb{R}^3$ which is transversal to the line $\{1, 1, x \mid x \in \mathbb{R}\} \subset \mathbb{R}^3$.

T. Consider $f(x) = (0, 0, x)$: vacuously transversal (as $f^{-1}\{(1,1,x) \mid x \in \mathbb{R}\} = \emptyset$)

(d) If $f : X \to Y$ is a submersion then f is transversal to Z for any manifold $Z \subset Y$.

True. $d_pf(T_pX) = T_pY$

so $d_pf(T_pX) + \text{anything} = T_pY$.

(e) A measure zero subset of S^1 consists of finitely many points.

F. Take any countable subset of S^1, say $\{(\cos q\pi, \sin q\pi) \mid q \in \mathbb{Q}\}$.

Countable union of measure 0 is measure 0.