Math 141 Midterm 2 and Final exam practice problems
Transversality, Sard's theorem, stability and homotopy, manifolds with boundary.

Note that the difficulty of these problems is not representative of the difficulty of the exam: some of these are quite tricky. As before, exam problems will be on the level of simplified homework problems.

1. (a) Let $E := W + \vec{v} \subset \mathbb{R}^n$ be a translation of a k-dimensional subspace $W \subset \mathbb{R}^n$ by a vector v which is orthogonal to W. Show that E is transversal to the sphere S^n if and only if $|v| \neq 1$.

S^n is the preimage of the regular value $1 \in \mathbb{R}$ under the function $G : \mathbb{R}^{n+1} \to \mathbb{R}$ with $G(x_1, \ldots, x_{n+1}) := \sum x_i^n$.

So by a result from the book/from class, S^n intersects a submanifold $Z \subset \mathbb{R}^{n+1}$ transversally if and only if 1 is a regular value for the function $g := G \mid Z : Z \to \mathbb{R}$. Let $Z = \vec{v} + W$. Then the restriction $g := G \mid Z$ has differential $d_p g = d_p G \mid_{T_p W}$, for $p \in Z$. Now $T_p \vec{v} + W = W$ (for all points p) and $d_p G = 2p^T$, where p is viewed as a vectical vector. So $d_p g = 2p^T \mid W$. This linear map to a one-dimensional space is surjective if and only if it has a nonzero value, equivalently if and only if W contains a vector which is *not* orthogonal to p.

Now we have three cases. First, if $|\vec{v}| > 1$, then $|\vec{v} + \vec{w}| = \sqrt{|\vec{v}|^2 + |\vec{w}|^2} > 1$, and so $S^n \cap Z = \emptyset$, implying transversality vacuously. Second, if $|\vec{v}| = 1$, then $\vec{v} \in S^n \cap Z$, and $d_p g = 2\vec{v}^T \mid W$, which is zero as $\vec{v}$ is (by assumption) orthogonal to W. Finally, assume $|\vec{v}| < 1$. Let $p \in S^1 \cap Z$. Then as $p \in Z$ we have $p = \vec{v} + \vec{w}$ for a vector $\vec{w} \in W$. As $p \in S^1$, we must have $|\vec{w}| = \sqrt{1 - |\vec{v}|^2} > 0$. In particular, $|\vec{w}| \neq 0$ and so viewing $\vec{w} \in W$ as a tangent vector in $T_p Z (= W)$, we have $dg(\vec{w}) = p^T(\vec{w}) = (\vec{v} + \vec{w}) \cdot \vec{w} = 0 + \vec{w} \cdot \vec{w} > 0$, so $d_p g$ is surjective for each $p \in Z \cap S^n$, and Z and S^n are transversal.

(b) For what pairs of fixed values a, b is it true that the plane $\{(x, y, z, t) \mid x = a, y = b\}$ transversal to the sphere $S^2 \times \{0\}$ in $\mathbb{R}^4$?

We could use the previous method, with $S^2 \times \{0\}$ cut out by the function $G : \mathbb{R}^4 \to \mathbb{R}^2$ given by $(x, y, z, t) \mapsto (x^2 + y^2 + z^2, t)$. Instead, I'll use a more direct method.

Let $Z = \{(x, y, z, t) \mid x = a, y = b\}$. Then $T_p Z = \langle \vec{z}, \vec{t} \rangle$ is the span of z, t. On the other hand, let $p \in S^2 \times \{0\}$. Then $p = (x, y, z, 0)$, with $(x, y, z) \in S^2$.

The tangent space $T_p(S^2 \times \{0\})$ is $T_{(x,y,z)}S^2 \times \{0\}$, and $T_{(x,y,z)}S^2$ is the two-dimensional space $(x,y,z)^\perp$, i.e. the subspace $\{(x',y',z',0)^T \in \mathbb{R}^4 \mid xx' + yy' + zz' = 0$. Note that $T_p(S^2 \times \{0\}) \subset \mathbb{R}^3 \times \{0\}$ and $\langle \vec{z}, \vec{t} \rangle$ span all of $\mathbb{R}^4$ if and only if the two-dimensional space $T_pS^2 = (x,y,z)^\perp \subset \mathbb{R}^3$ together with $\vec{z}$ span all of $\mathbb{R}^3$. This is true if and only if $\vec{z}$ is not in $(x,y,z)^\perp$, i.e. $(x,y,z) \neq (x,y,0)$. So $Z \pitchfork S^2 \times \{0\}$ if and only if $Z \cap S^2 \times \{0\}$, does not contain vectors of the form $(a,b,0,t)$, i.e. if and only if $a^2 + b^2 \neq 1$.

2. (a) Assume that $B \subset X$ is a pair of (boundaryless) manifolds, with X of dimension n and B of dimension $n-1$. Prove that for any $b \in B$, there is a pair V, V' of manifolds with boundary such that the union $V \cup V'$ is an open neighborhood of b in X, but V, V' only intersect along B.

By the local embedding theorem, near each $p \in B \subset X$, there exists a pair of charts $\psi_X : U_X \to N_X, \psi_B : U_B \to N_B$, diffeomorphisms from opens $U_X \subset \mathbb{R}^n$ and $U_B \subset \mathbb{R}^{n-1}$ in Euclidean space to neighborhoods of p, such that $N_B = N_X \cap B$,[1] and we have $p = \psi_X(0) = \psi_B(0)$, and (for i_B the inclusion $B \subset X$) we have $\psi_X^{-1} i_B \psi_B = i_{n-1}^n$ for I_{n-1}^n the standard embedding from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n$. Set $V := \psi_X(H^+ \cap U)$ and $V' := \psi_X(H^- \cap U)$ (for $H^+ = \{(x_1, \ldots, x_n) \mid x_n > 0\}, H^- = \{(x_1, \ldots, x_n) \mid x_n < 0\}$ the half-spaces. Then V, V' are diffeomorphic to open subsets of $H^\pm$, therefore n-dimensional manifolds with boundary, and $V \cap V' = \psi_X(H^+ \cap H^-)$, satisfying the requirments of the problem.

(b) Assume $X = S^2$ and $B = S^1 \times \{0\}$ (the equator). Show that there is (globally) a pair V, V' of manifolds with boundary such that $V \cup V' = X$ and $V \cap V' = B$.

Take $V = S^2_{z \geq 0}, V' = S^2_{z \leq 0}$, the top and bottom hemispheres.

(c) Is this true for any submanifold B of codimension one in a manifold X? Hint: the open Möbius strip is a manifold (without boundary).

No. Consider the circle S^1 inside the open Möbius strip, M. Assume that there exist V, V' as above. Then as $V \cap V' = S^1$, note that S^1 must be in the boundary of both V and V' (if there is some point $p \in V' \cap \mathring{V}$ then as V is two-dimensional, its interior $\mathring{V}$ is open in M, and so $V' \cap \mathring{V}$ contains a nonempty open in V', which for dimension reasons cannot be contained in S^1).

So the interiors $\mathring{V}, \mathring{V}'$ are disjoint (have intersection $\emptyset$) and the unions of the interiors $V \cup V'$ must be $M \setminus S^1$. Since $M \setminus S^1$ is connected, the only way this can happen is if one of $\mathring{V}, \mathring{V}'$ is empty. But any nonempty manifold with boundary has nonempty interior, so $V' = \emptyset$, contradicting $V \cap V' = S^1$.

3. (a) We classified connected and compact one-dimensional manifolds with boundary up to diffeomorphism. Can you classify all connected manifolds with boundary in $\mathbb{R}^1$? (Not up to diffeomorphism.) Which ones are compact?

[1] if $N_B \subset N_X$ but is not all of $N_X \cap B$, just throw away from N_X all points which are in B but not in N_B: this is a closed set.

Let $X \subset \mathbb{R}$ be a connected manifold with boundary. Then $\mathring{X}$ is a connected manifold inside $\mathbb{R}$ of dimension 1, therefore an open connected subset of $\mathbb{R}$. All open connected subsets in $\mathbb{R}$ are of the form $U = (a, b), U = (a, \infty), U = (-\infty, b), U = (-\infty, \infty)$ for some $a < b \in \mathbb{R}$. Note that for any manifold with boundary X we have (in the topology of X), the topological closure $\overline{\left(\mathring{X}\right)} = X$ (see this locally: it is true for the half-space H). Now let $Y := \overline{\mathring{X}}_{\mathbb{R}}$ be the topological closure of the manifold $\mathring{X}$ in $\mathbb{R}$. Then Y is closed in the ambient space, so $X \cap Y$ is closed in X and contains $\mathring{X}$, therefore $X \cap Y = X$, and so $X \subset Y$. This means that for some connected open $U \subset \mathbb{R}$ (as above), we must have $U \subset X \subset \bar{U}$. Any choice of such a set is evidently a manifold with boundary. The choices are as follows:

1. For $\mathring{X} = (a, b)$ we can add zero, one or both boundary points, resulting in options $(a, b), [a, b), (a, b]$ and $[a, b]$.
2. For $\mathring{X} = (a, \infty)$ or $(-\infty, b)$ we can add zero or one boundary points resulting in options $(a, \infty), [a, \infty), (-\infty, b), (-\infty, b]$.
3. For $\mathring{X} = (-\infty, \infty)$, there are no boundary points we could add, so X must be $(-\infty, \infty)$ (empty boundary).

(b) Same question, in S^1.

Let $X \subset S^1$ be such a manifold. We have two options. Either $X = S^1$, or there is some point $p_0 \in S^1 \setminus X$. In the latter case, we have $X \subset S^1 \setminus p$. Now $S^1 \setminus p_0$ is diffeomorphic to the open interval $(\theta, \theta + 2\pi)$ under the winding map $\alpha : \mathbb{R} \to S^1$, for some $a \in \mathbb{R}$. Therefore manifolds with boundary in S^1 are either S^1 or $\alpha(X)$ for X a manifold with boundary in $\mathbb{R}$ contained in some $(\theta, \theta + 2\pi)$. In other words we have the following possibilities: S^1 (no boundary) or $\alpha((a, b))$ for $b - a \leq 2\pi$, or $\alpha((a, b]), \alpha([a, b)), \alpha([a, b])$ for $b - a < 2\pi$.

4. Construct a homotopy from the inclusion $i : S^1 \to D^2$ of the circle in the two-disk to the map $z : S^1 \to D^2$ sending each point to $(0, 0)$.

Define $i_t(p) = p \cdot (1 - t)$ for $p \in S^1$. Note that for $t \in [0, 1]$ we have $i_t(p) \in D$, so we can define a map $h : S^1 \times I \to D$ by $h(x, t) := i_t(p)$. This map is evidently smooth, and it is a homotopy between $i_0 = i$ and $i_1 = z$.

5. (a) Show that the property of a smooth function $I \to \mathbb{R}$ to have everywhere negative derivative is stable[2].

Assume $f_t : I \to \mathbb{R}$ is a smoothly varying homotopy of such functions, with f_0 having everywhere negative derivative. Since I is compact, the property of such a function being an immersion, equivalently local isomorphism is stable (from the book), so for some $\epsilon > 0$, all f_t for $t < \epsilon$ are immersions. Therefore for these t, we have $f_t'(x) \neq 0$ for any $x \in I$. As $f_0'(x) < 0$, by the intermediate value theorem we must have $f_t'(x) < 0$ for all $x \in I$ and $t < \epsilon$.

[2]original question was for map $\mathbb{R} \to \mathbb{R}$, which is false

(b) Show that this property is not homotopy invariant. Consider the homotopy $h(x,t) = f_t(x)$ for $f_t(x) = (t - 1/2) \cdot x$. Then at $f_0(x)$ is the function $x \mapsto -x/2$ which has negative derivative but $f_1(x) = x/2$ certainly does not.

(c) If X is a compact manifold with boundary, the property of a smooth map $f : X \to Y$ being a diffeomorphism[3] onto its image (i.e., an embedding) is stable (you do not need to prove this). Show that this is not necessarily true if the maps are only required to be continuous. Hint: take $X = [0,1]$, $Y = \mathbb{R}$ and $f = i$ the standard embedding. Can you deform it such a way that it is no longer diffeomorphic to its image?

Consider the continuous map $i : I \to \mathbb{R}$ given by $x \mapsto x$. This map is a diffeomorphism of I to $[0,1] \subset \mathbb{R}$. Now let $f_t := |x - t|$ for $t \in I$. Then $f_0 = i$ but for any $\epsilon > 0$, the function f_ϵ is not injective (for example $f_\epsilon(-\epsilon/2) = f_\epsilon(\epsilon/2)$.)

6. (a) Give an example of a function $f : X \to Y$ whose critical *points* do not have measure zero (and explain why this is the case).

Consider the zero map $z : \mathbb{R} \to \mathbb{R}$, with $t \mapsto 0$. Then every point of $\mathbb{R}$ is critical (but 0 is the only critical value). Alternatively, consider the inclusion $i : S^1 \to \mathbb{R}^2$. Then every point of S^1 is critical (by dimension reasons, i cannot be a submersion), but any $q \in \mathbb{R}^2 \setminus S^1$ is a regular value, vacuously.

(b) Suppose that $f : X \to \mathbb{R}$ is a smooth *and proper* map from a one-dimensional manifold *with boundary*[4]. Assume that f has finitely many critical values. Show that the number of preimage points of a regular value is bounded by a finite number.

Note that this is a difficult problem, and not representative of examp problems. Let $C \subset \mathbb{R}$ be the finite set of critical values. Arranging them in order, write $C = \{c_1, c_2, \ldots, c_k\}$ with $c_1 < c_2 < \cdots < c_k$. Let $I_0 = (-\infty, c_1)$, and let $I_1 = (c_1, c_2)$, $I_2 = (c_2, c_3), \ldots, I_{k-1} = (c_{k-1}, c_k)$ and $I_k = (c_k, \infty)$. Let $a < b$ be a pair of points in a given interval I_ℓ. Then all points of the closed interval $[a,b]$ are regular points of f, so $f^{-1}([a,b])$ is a one-dimensional manifold with boundary. It is compact, since f is proper and $[a,b]$ is compact. So $f^{-1}([a,b])$ is a finite union of closed intervals and circles. A circle cannot map to (a,b) without critical points (the maximum and minimum are critical), so $f^{-1}([a,b])$ is a union of closed intervals. The boundary $\partial f^{-1}([a,b]) = f^{-1}(\partial[a,b]) = f^{-1}(a) \sqcup f^{-1}(b)$.

A submersion from a closed interval to $[a,b]$ must be monotone, therefore the two boundary points of each connected component of $f^{-1}([a,b])$ map to two different boudnary points. Therefore, $f^{-1}(a)$ and $f^{-1}(b)$ are finite sets of the same finite cardinality (equal to the number of intervals in $f^{-1}[a,b]$).) Let n_ℓ be the number of boundary points of some point t in the interval I_ℓ, (this is finite as points $t \in I_\ell$ are regular and f is proper). Then any other point $t' \in I_\ell$ also has n_ℓ preimage points, and as the union $I_0 \cup I_1 \cup \cdots \cup I_n \subset \mathbb{R}$ is the complement to the set of critical values, the maximum number of critical points is $\sup(n_0, n_1, \ldots, n_k)$, a finite number.

[3] original problem had a typo and said "diffeomorphism"

[4] originally, the words "proper" and "with boundary" were not mentioned

7. Which of the following are manifolds with boundary? Give a picture or a very brief explanation.

(a) The ray, $[0, \infty)$**?** Yes: it is a one-dimensional half-space.

(b) The rectangle $X = [0, a] \times [0, b]$? No. This is a rectangle. Its interior points have an open neighborhood diffeomorphic to $\mathbb{R}^2$ but its exterior points do not (as they have a tangent direction that cannot be extended to the image of an interval). But its exterior (the boundary of a rectangle) is not a smooth manifold, which would be the case if X were a manifold with boundary.

(c) **The cross,** $\{(x, y) \mid xy = 0\} \subset \mathbb{R}^2$? No. The midpoint $p = (0, 0)$ has no neighborhood diffeomorphic to either an open or a closed interval (for example, if you remove p from any neighborhood you get at least four connected components).

(d) **The area under a graph,** $\{(x, y) \mid x \leq f(y)\} \subset \mathbb{R}^2$, **for** f **a smooth function?** Yes. The map $(x, y) \mapsto (x, y - f(x))$ provides a chart to the lower half-plane.

(e) The set $X = \{(x, y, z) \mid x^2 + y^2 \leq z^2$ **in** $\mathbb{R}^3$**?** No. This is a filled-in union of two opposite cones, and its boundary is the union of two empty cones. The cone point is a singularity of the boundary. **Here's a more careful proof:** Assume there was a chart ψ from H^3 to a neighborhood of $p = (0, 0, 0)$, taking $0 \in H^3$ to p. Let $A = d_0\psi$ (an injective, therefore invertible matrix). Let $\vec{v} = A^{-1}(0, 0, 1)$. Then either $\vec{v} \in H$ or $-\vec{v} \in H$. Define the map $\gamma : I \to X$ defined by

$$\gamma(t) := \psi(\pm t\vec{v})$$

(sign chosen so that $\pm\vec{v} \in H$). This map has differential $\gamma'(0) = (\pm 1, 0, 0)$,. But this means that for sufficiently small ϵ, we have $gamma(\epsilon) = (\epsilon, 0, 0) + O(\epsilon^2)$, and so for $(x, y, z) = \gamma(\epsilon)$ we have $x^2 = \epsilon^2 + O(\epsilon^3)$ is certainly larger than $z^2 = O(\epsilon^4)$, so $\gamma(\epsilon)$ is not in X, contradiction.

8. True/false.

(a) If X **is compact, a function** $f : X \to \mathbb{R}$ **has finitely many critical values.** False. We only know that the set of critical values has measure zero. (For an example, consider the function $f(x) := \sin(1/x) \cdot e^{-1/x^2}$ extended by $f(0) = 0$ from $[0, 1]$ to $\mathbb{R}$).

(b) There is a one-dimensional manifold Y **and a map** $f : D^2 \to Y$ **with** $S^1 \subset D^2$ **mapping injectively.** False. If S^1 maps injectively, then ∂f is non-constant, so there are points where the differential of ∂f is nonzero, thus invertible, hence (by local diffeomorphism theorem), the image of δf contains an open neighborhood V of Y. This means $f(S^1) \subset Y$ contains a regular point (regular points are dense by Sard), say $q = f(p) \in Y$, for $p \in S$. The preimage of q is a one-dimensional submanifold with boundary $Q \subset D^2$, compact as D^2 is compact. Its boundary is $\partial Q = Q \cap S^1 = \{p\}$, as ∂f is injective. But a compact one-dimensional manifold has even number of boundary points, contradiction.

(c) The union of measure zero sets is measure zero. True, from class/book.

(d) The product of a manifold with boundary and a manifold without boundary is a manifold with boundary. True, from class/book (or see this by looking locally and using that $H^m \times \mathbb{R}^n \cong H^{m+n}$).

(e) S^2 is a manifold with boundary. True. The boundary happens to be empty.

9. Let $X \subset \mathbb{R}^2$ be a one-dimensional manifold. Show that there exists a number t such that the set of points $(x, y) \in X$ with $x - y = t$ is discrete.

Let $f : \mathbb{R} \to \mathbb{R}^2$ be given by $f(r) = (r, r)$. By the translation transversality theorem, there is a vector $v = (a, b) \in \mathbb{R}^2$ with the function $f_v : r \mapsto (r+a, r+b)$ transversal to X. By the preimage theorem, for these a, b the set of points $\{r \in \mathbb{R} \mid (r + a, r + b)\} \in X$ is a zero-dimensional manifold, equivalently, a discrete set of points. We conclude by setting $t = a - b$.

Note. An alternative proof is to define the function $F : \mathbb{R}^2 \to \mathbb{R}$ with $F(x, y) = x - y$, set $f = F\,|_X$, and set t to be a regular value of f.

10. Show that if $A, B \subset X$ are subsets (not necessarily closed) such that A has measure 0 and B does not have measure zero then the complement of $B \setminus (B \cap A)$ of A in B does not have measure zero (note: do not try to work with the measure of B: rather, use proof by contradiction).

Set $A' = B \cap A$. Since $A' \subset A$, it has measure zero. Assume $B \setminus A'$ has measure zero. Then $B = A' \cup (B \setminus A')$ is the union of two sets of measure zero, hence has measure zero. Contradiction.