Math 141 Midterm 2 and Final exam practice problems
Transversality, Sard's theorem, stability and homotopy, manifolds with boundary.

1. (a) Let $E := W + \vec{v} \subset \mathbb{R}^n$ be a translation of a k-dimensional subspace $W \subset \mathbb{R}^n$ by a vector v which is orthogonal to W. Show that E is transversal to the sphere S^n if and only if $|v| \neq 1$.

(b) For what pairs of fixed values a, b is it true that the plane $\{(x, y, z, t) \mid x = a, y = b\}$ transversal to the sphere $S^2 \times \{0\}$ in $\mathbb{R}^4$?

2. (a) Assume that $B \subset X$ is a pair of (boundaryless) manifolds, with X of dimension n and B of dimension $n-1$. Prove that for any $b \in B$, there is a pair V, V' of manifolds with boundary such that the union $V \cup V'$ is an open neighborhood of b in X, but V, V' only intersect along B.

(b) Assume $X = S^2$ and $B = S^1 \times \{0\}$ (the equator). Show that there is (globally) a pair V, V' of manifolds with boundary such that $V \cup V' = X$ and $V \cap V' = B$.

(c) Is this true for any submanifold B of codimension one in a manifold X? Hint: the open Möbius strip is a manifold (without boundary).

3. (a) We classified connected and compact one-dimensional manifolds with boundary up to diffeomorphism. Can you classify all connected manifolds with boundary in $\mathbb{R}^1$? (Not up to diffeomorphism.) Which ones are compact?

(b) Same question, in S^1.

4. Construct a homotopy from the inclusion $i : S^1 \to D^2$ of the circle in the two-disk to the map $z : S^1 \to D^2$ sending each point to $(0, 0)$.

5. (a) Show that the property of a smooth function $I \to \mathbb{R}$ to have everywhere negative derivative is stable[1].

(b) Show that this property is not homotopy invariant.

[1] original question was for map $\mathbb{R} \to \mathbb{R}$, which is false

(c) If X is a compact manifold with boundary, the property of a smooth map $f : X \to Y$ being a diffeomorphism[2] onto its image (i.e., an embedding) is stable (you do not need to prove this). Show that this is not necessarily true if the maps are only required to be continuous. Hint: take $X = [0, 1]$, $Y = \mathbb{R}$ and $f = i$ the standard embedding. Can you deform it such a way that it is no longer diffeomorphic to its image?

6. (a) Give an example of a function $f : X \to Y$ whose critical *points* do not have measure zero (and explain why this is the case).

(b) Suppose that $f : X \to \mathbb{R}$ is a smooth *and proper* map from a one-dimensional manifold *with boundary*[3]. Assume that f has finitely many critical values. Show that the number of preimage points of a regular value is bounded by a finite number.

7. Which of the following are manifolds with boundary? Give a picture or a very brief explanation.

(a) The ray, $[0, \infty)$?

(b) The rectangle $[0, a] \times [0, b]$?

(c) The cross, $\{(x, y) \mid xy = 0\} \subset \mathbb{R}^2$?

(d) The area under a graph, $\{(x, y) \mid x \leq f(y)\} \subset \mathbb{R}^2$, for f a smooth function?

(e) The set $\{(x, y, z) \mid x^2 + y^2 \leq z^2$ in $\mathbb{R}^3$?

8. True/false.

(a) If X is compact, a function $f : X \to \mathbb{R}$ has finitely many critical values.

(b) There is a one-dimensional manifold Y and a map $D^2 \to Y$ with $S^1 \subset D^2$ mapping injectively.

(c) The union of measure zero sets is measure zero.

(d) The product of a manifold with boundary and a manifold without boundary is a manifold with boundary.

(e) S^2 is a manifold with boundary.

9. Let $X \subset \mathbb{R}^2$ be a one-dimensional manifold. Show that there exists a number t such that the set of points $(x, y) \in X$ with $x - y = t$ is discrete.

10. Show that if $A, B \subset X$ are subsets (not necessarily closed) such that A has measure 0 and B does not have measure zero then the complement of $B \backslash (B \cap A)$ of A in B does not have measure zero (note: do not try to work with the measure of B: rather, use proof by contradiction).

[2] original problem had a typo and said "diffeomorphism"

[3] originally, the words "proper" and "with boundary" were not mentioned