Math 141 | Midterm 1 | October 15, 2019

Additional sheets available (write your name on any additional sheets!!)

Name: ______________________________

1. Briefly define the following terms.
   (a) A smooth manifold.

   (b) A submersion.

   (c) A compact set $X \subset \mathbb{R}^n$ (equivalently, a set such that the subspace topology is compact). Either of our two definitions are ok.

   (d) An embedding (either the class or the book definition, which we proved equivalent, are ok).

Extra space

2. (a) Find all critical values of the function

$$F : \mathbb{R}^3 \to \mathbb{R}$$

with $F(x, y, z) = xyz$.

(b) Give an example of a regular value $q \in \mathbb{R}$, describe the preimage $F^{-1}(q)$ and find the tangent space of every point $p \in F^{-1}(q)$.

(c) Let $g : S^2 \to \mathbb{R}$ be the function given by $g(x, y, z) = x + y$ (for $(x, y, z) \in S^2$). Find all critical values of g. You may use without proof that $T_p(S^s) = \{\vec{v} \in \mathbb{R}^3 \mid \vec{v} \cdot p = 0\}$, where $p \in S^2$ is a point, viewed as a vector: in other words, $T_p(S^2)$ is the space of all vectors in $\mathbb{R}^3$ orthogonal to p.

Extra space

3. Let X be a manifold of dimension n and Y of dimension m. Informally, the local immersion theorem states that for any map $f : X \to Y$ of manifolds which is an immersion at $p \in X$, there are neighborhoods N_p of p and N_q of $q = f(p)$ such that f is equivalent to the linear map $i : U \to V$ for $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ opens in Euclidean space and $i : U \to V$ defined by $i(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, 0, \ldots, 0)$.

   (a) Make the above informal statement rigorous to give a concrete statement of the local immersion theorem. (Hint: you will probably want to consider a pair of charts, a.k.a. parametrizations, for N_p and N_q.)[1]

   (b) Let $i : S^1 \to S^2$ be the embedding of the equator, with $i(x, y) = (x, y, 0)$ for $(x, y) \in S^1$. Show that i is an immersion.

   (c) For the point $p = (1, 0)$ and $q = i(P) = (1, 0, 0)$, find a pair of neighborhoods of p, q and charts ψ_p, ψ_q which verify the statement of the local immersion theorem in this case.

[1] In class I gave a simplified statement with $U = \mathbb{R}^n, V = \mathbb{R}^m$. You may give a rigorous version of that statement instead if you prefer.

Extra space

4. Let

$$X = \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}.$$

(a) Show that $X \cap GL_2$ is a Lie group. (You do not need to prove that X is a manifold, so the "manifold" part of Lie group should be almost obvious. You need to verify closure under $\cdot$ and inverse.)

(b) Show that the multiplication map $\mu : X \times X \to X$ (taking (M, N) to their product MN) is a submersion at $p = (I, I) \in X \times X$.

(c) Show that for the matrix $M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in X$, the map μ is not a submersion at (M, M). Deduce that M is a critical value of the map $\mu : X \times X \to X$ as above.

Extra space

5. True/false (and a one-sentence explanation).

   (a) A differentiable map is smooth.

   (b) There exists an immersion from $\mathbb{R}$ to S^2.

   (c) There exists a submersion from $\mathbb{R}$ to S^2.

   (d) If $d_p f$ is surjective then $f(p)$ is a regular value.

   (e) The preimage of a regular value is a manifold.

Extra space