Math 141 Midterm 1 October 15, 2019

Additional sheets available (write your name on any additional sheets!!)

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1. Briefly define the following terms.

(a) A smooth manifold.

$X \subset \mathbb{R}^N$ | each $p \in X$ has an open neighbd $\cancel{M}$ $\subset X$ diffeomorphic to an open $U \subset \mathbb{R}^n$ in Euclidean space.

(b) A submersion.

A map $f : X \to Y$ between manifolds such that $\forall p \in X$, $d_p f : T_p X \to T_{f(p)} Y$ is surjective

(c) A compact set $X \subset \mathbb{R}^n$ (equivalently, a set such that the subspace topology is compact). Either of our two definitions are ok.

option 1: A set $X \subset \mathbb{R}^n$ such that for any ~~union~~ set $\mathcal{U} \subset \mathcal{O}_X$ of opens in X (in subs. top.), if $\bigcup_{u \in \mathcal{U}} u = X$ then there is a finite subset $u_1, \dots, u_N \in \mathcal{U}$ which covers X (i.e. $\bigcup_{i=1}^{N} u_i = X$)

Option 2 $X \subset \mathbb{R}^n$ is compact if it is closed & bounded, i.e. contained in a finite ball B.

(d) An embedding (either the class or the book definition, which we proved equivalent, are ok).

option 1 An immersion $f : X \to Y$ which is injective and ~~compact~~ proper, i.e. $K \subset Y$ compact $\Rightarrow f^{-1}(K) \subset X$ compact.

option 2 A map $f : X \to Y$ s.t. $f : X \to f(X)$ is a diffeo b/w X & $f(X) \subset Y$.

2. (a) Find all critical values of the function

$$F : \mathbb{R}^3 \to \mathbb{R}$$

with $F(x, y, z) = xyz$.

multivariable calculus problem. For $p=(x,y,z)$

$d_pF = (yz, xz, xy)$. Surjective $\iff$ nonzero, since mapping to $\mathbb{R}$.

$d_pF = 0 \iff (yz, xz, xy) = 0 \iff$ at least two out of $\{x, y, z\}$ are 0. In this case $F(x,y,z) = 0$, so $q=0$ only critical value.

(b) Give an example of a regular value $q \in \mathbb{R}$, describe the preimage $F^{-1}(q)$ and find the tangent space of every point $p \in F^{-1}(q)$.

Take $q = 1$.

$F^{-1}(q) = \{(x, y, \frac{1}{xy}) \mid x, y \neq 0\}$.

For $p = (x, y, \frac{1}{xy})$, by preimage theorem have

$T_pF^{-1}(q) = \operatorname{Ker} d_pF = \operatorname{Ker}(yz, xz, xy) =$

$\operatorname{Ker}(\frac{1}{x}, \frac{1}{y}, xy) = \langle (x, -y, 0), (0, xy^2, -1) \rangle$.

(c) Let $g : S^2 \to \mathbb{R}$ be the function given by $g(x, y, z) = x + y$ (for $(x, y, z) \in S^2$). Find all critical values of g. You may use without proof that $T_p(S^s) = \{\vec{v} \in \mathbb{R}^3 \mid \vec{v} \cdot p = 0\}$, where $p \in S^2$ is a point, viewed as a vector: in other words, $T_p(S^2)$ is the space of all vectors in $\mathbb{R}^3$ orthogonal to p.

Let $\tilde{g} : \mathbb{R}^3 \to \mathbb{R}$ given by $\tilde{g}(x,y,z) = x+y$.

$d_{(x,y,z)}\tilde{g} = \operatorname{Jac} \tilde{g}(x,y,z) = (1, 1, 0)$.

Let $i : S^2 \to \mathbb{R}^3$ be inclusion. By HW,

$d_i : T_pS^2 \to T_p\mathbb{R}^3 = \mathbb{R}^3$ is inclusion $i_T : T_pS^2 \subset \mathbb{R}^3$ of sub-vector space. By chain rule

$d_pg = d_p(\tilde{g} \circ i) = d_p\tilde{g} \circ d_pi = d_p\tilde{g} \circ i_T =$

$\tilde{g}|_{T_pS^2}$. By descr. of T_pS^2 given, we see

$d_pg = 0 \iff d_p\tilde{g}$ is proportional to p [over]

Extra space

so: $p = \lambda \cdot (1,1,0)$ (some constant λ)

$\Rightarrow p \in \{\pm \sqrt{2}/2 \cdot (1,1,0)\}$.

The critical values are

$g(p)$ for these points p, i.e.

$g(\frac{\sqrt{2}}{2} \cdot (1,1,0)) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \in \mathbb{R}$ and

$g(-\frac{\sqrt{2}}{2}(1,1,0)) = -\frac{\sqrt{2}}{2} + (-\frac{\sqrt{2}}{2}) = -\sqrt{2} \in \mathbb{R}$.

3. Let X be a manifold of dimension n and Y of dimension m. Informally, the local immersion theorem states that for any map $f : X \to Y$ of manifolds which is an immersion at $p \in X$, there are neighborhoods N_p of p and N_q of $q = f(p)$ such that f is equivalent to the linear map $i : U \to V$ for $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ opens in Euclidean space and $i : U \to V$ defined by $i(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, 0, \dots, 0)$.

(a) Make the above informal statement rigorous to give a concrete statement of the local immersion theorem. (Hint: you will probably want to consider a pair of charts, a.k.a. parametrizations, for N_p and N_q.)[1]

Theorem. If $f : X \to Y$ is an immersion then for any $p \in X$,

There is a neighborhood $N_p \subset X$ of p and $N_q \subset Y$ of $q = f(p)$ ~~such that~~ satisfying $f(N_p) \subset N_q$ and charts $\varphi_p : U \to N_p$, $\varphi_q : V \to N_q$ (diffeos w/ Euclidean opens) such that $g := \varphi_q^{-1} \circ f \circ \varphi_p : U \to V$ is given by $g(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, 0, \dots, 0)$. [see diagram]

diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ N_p \arrow[u, phantom, "\cup"] \arrow[r, "f"] & N_q \arrow[u, phantom, "\cup"] \\ U \arrow[u, "\varphi_p"] \arrow[r, "g = i"] & V \arrow[u, "\varphi_q"'] \end{tikzcd}$$

(b) Let $i : S^1 \to S^2$ be the embedding of the equator, with $i(x, y) = (x, y, 0)$ for $(x, y) \in S^1$. Show that i is an immersion.

Option 1 $T_p(S^1) = \langle (-y, x) \rangle$ (for $p = (x, y)$)

$di(-y, x) = (-y, x, 0)$ [implicitly using that $i = \tilde{i}|_{S^1}$ for $\tilde{i} : \mathbb{R}^2 \to \mathbb{R}^3$ taking (x, y) to $(x, y, 0)$]

Nonzero map from 1d-vector space $\Rightarrow$ injective $\Rightarrow$ immersion.

~~Option 2 Choose a chart~~

Option 2 charts: Say $p = (\cos\theta, \sin\theta)$. standard chart:
$\psi(t) = (\cos t, \sin t)$ for $t \in (\theta - \pi, \theta + \pi)$.
$d_t\, i \circ \psi = d_t(\cos t, \sin t, 0) = (\cancel{-\sin t}, \cos t) \neq 0 \Rightarrow$ immers.

(c) For the point $p = (1, 0)$ and $q = f(P) = (1, 0, 0)$, find a pair of neighborhoods of p, q and charts ψ_p, ψ_q which verify the statement of the local immersion theorem in this case.

rename i by f.

Many possibilities. Here is one: Take $N_p \subset S^1 = \{(x, y) \mid x > 0\}$, $N_q \subset S^2 = \{(x, y, z) \mid x > 0\}$.

$\varphi_p(x) := (\sqrt{1 - x^2}, x)$ ~~for $|x| < 1$ (set $U = (-1, 1)$)~~

$\varphi_q(x, y) := (\sqrt{1 - x^2 - y^2}, x, y)$ for $x^2 + y^2 < 1$ (set $V = D^2_1$, open disk)

Check:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\varphi_p"] \arrow[r, "i"] & V \arrow[u, "\varphi_q"'] \end{tikzcd}$$

commutative: indeed, $\varphi_q \circ i(x) = \varphi_q(x, 0) = (\sqrt{1 - x^2}, x, 0)$ and

$f \circ \varphi_p(x) = f(\sqrt{1 - x^2}, x) = (\sqrt{1 - x^2}, x, 0)$

so $f \circ \varphi_p = \varphi_q \circ i$. Q.E.D.

[1] In class I gave a simplified statement with $U = \mathbb{R}^n, V = \mathbb{R}^m$. You may give a rigorous version of that statement instead if you prefer.

4. Let

$$X = \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}.$$

Write $G = X \cap GL_2$.

(a) Show that $X \cap GL_2$ is a Lie group. (You do not need to prove that X is a manifold, so the "manifold" part of Lie group should be almost obvious. You need to verify closure under $\cdot$ and inverse.)

- Manifold: $X \cap GL_2 =$ linear subspace $\cap$ open $\subset \mathbb{R}^4 \Rightarrow$ manifold.

$\begin{vmatrix} 1 & a \\ 0 & b \end{vmatrix} = 1 \cdot b - 0 \cdot a = b$, so $\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \in GL_2$ iff $b \neq 0$.

- (optional) show $I \in X \cap GL_2$: clear.
- closure: $\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & a' \\ 0 & b' \end{pmatrix} = \begin{pmatrix} 1 & a' + ab' \\ 0 & bb' \end{pmatrix} \in X \cap GL_2$, assuming $b, b' \neq 0$
- inverse: $\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a/b \\ 0 & b^{-1} \end{pmatrix} \in X \cap GL_2$.

(b) Show that the multiplication map $\mu : X \times X \to X$ (taking (M, N) to their product MN) is a submersion at $p = (I, I) \in X \times X$.

Use chart $\psi : (a, b, a', b') \mapsto \left(\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}, \begin{pmatrix} 1 & a' \\ 0 & b' \end{pmatrix} \right)$ from $\{(a, b, a', b') \in \mathbb{R}^4 \mid b, b' \neq 0\}$ to $G \times G$. Then

$d\, \mu \circ \psi = \text{Jac} \begin{pmatrix} 1 & a' + ab' \\ 0 & bb' \end{pmatrix} = \left(\begin{pmatrix} 0 & b' \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & b' \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \right)$

Since $b, b' \neq 0$ Already first two partials $\begin{pmatrix} 0 & b' \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & b' \end{pmatrix}$ span

Alt: use hw (briefly recollect proof for full credit)

(c) Show that for the matrix $M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in X$, the map μ is not a submersion at (M, M). Deduce that M is a critical value of the map $\mu : X \times X \to X$ as above.

From above we have partial deriv's

$$\begin{pmatrix} 0 & b' \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & b' \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$$

for $\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1 & a' \\ 0 & b' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. So $b = b' = 0$

& only nonzero partial derivative is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow$ image of $d_{(M,M)}\mu$ is 1-dim whereas $T_M X$ is 2-dim $\Rightarrow$ not surjective on tangents $\Rightarrow$ not subm.

[over]

Extra space

so $\mu(M, M) = M$ is a critical value.

5. True/false (and a one-sentence explanation).

(a) A differentiable map is smooth.

Not necessarily, higher derivatives can be undefined (ex: $f(x) = x^{4/3}$)

(b) There exists an immersion from $\mathbb{R}$ to S^2.

True. Ex: $f(t) = (\cos t, \sin t, 0)$.

(c) There exists a submersion from $\mathbb{R}$ to S^2.

false: $\underset{\overset{\shortparallel}{1}}{\dim \mathbb{R}} < \underset{\overset{\shortparallel}{2}}{\dim S^2}$)

so cannot have submersion.

(d) If $d_p f$ is surjective then $f(p)$ is a regular value.

False. Needs to be ~~regular value~~ subm. at each preimage of $f(p)$, not only p.

Ex: $F(x,y,z) = xyz$ is a submersion at $(1,1,0)$ but $F(1,1,0) = 0$ is critical.

(e) The preimage of a regular value is a manifold.

T. This is the preimage theorem.