# Math 141 Midterm 1 and Final exam practice
# Topology, immersions, submersions, Lie groups.

**1. Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous. Show that its graph $\Gamma(f) \subset \mathbb{R}^{m+n}$, defined by $\Gamma(f) := \{(x, y) \mid f(x) = y\} \subset \mathbb{R}^m \times \mathbb{R}^n$ is closed.**
Let $F : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be the function defined as $F(x, y) := y - f(x)$. Then since $f$ is continuous, $F$ is also continuous, and therefore the preimage of a closed set under $F$ is closed. We are done by observing $\Gamma = F^{-1}(\{\vec{0}\})$.

**2. Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.**
**Short proof.** Assume for the sake of contradiction that a homeomorphism $f : \mathbb{R} \to \mathbb{R}^2$. Let $\mathbb{R}^* := \mathbb{R} \setminus 0$, and $U := \mathbb{R}^2 \setminus f(0)$. Let $f^* : \mathbb{R}^* \to U$ be the restriction of $f$. Since $f$ induces a bijection on opens $\mathcal{O}_\mathbb{R} \to \mathcal{O}_{\mathbb{R}^2}$, it must do the same after restricting to the complement of a point, so $f^* : \mathcal{O}_{\mathbb{R}^*} \to \mathcal{O}_U$ is a bijection, and $f^*$ is a homeomorphism. $\mathbb{R}^*$ is disconnected but $\mathbb{R}^2$ without a point is easily seen to be path-connected (and therefore connected). Contradiction.

**Long proof.** Same, but check more carefully that $f^*$ is a homeomorphism and that $V$ is path-connected:
*Checking that $f^*$ is a homeomorphism.* First prove a basic topology lemma:

**Lemma 1.** *Suppose $f : X \to Y$ is a continuous function and $A \subset X$ and $B \subset Y$ are subsets such that $f(A) \subset B$. Then the restricted function $f_{res} : A \to B$ is continuous for the subset topologies of $A$, $B$.*

*Proof.* Suppose $V \subset B$ is open. Then there exists $U \subset Y$ with $V = U \cap B$. Now $f_{res}^{-1}(B) = f^{-1}(U) \cap A$, open for the subset topology $A$ by continuity of $f$. □

Now apply the lemma to $A = \mathbb{R}^*, B = \mathbb{R}^2 \setminus f(0)$ to see that $f^*$ is continuous, and with $A, B$ flipped to see $(f^*)^{-1}$ is continuous. □

*Checking that $\mathbb{R}^2 \setminus f(0)$ is path-connected.* Let $P_0 = f(0) \in \mathbb{R}^2$. Suppose $P_1, P_2 \in \mathbb{R}^2 \setminus P$ are two distinct points in the complement to $P_0$. Consider two (of the infinitely many) distinct circular arcs $\gamma, \gamma'$ connecting $P_1$ to $P_2$ in $\mathbb{R}^2$. Then $\gamma, \gamma'$ do not intersect (except at the endpoints), and so at most one of $\gamma, \gamma'$ may contain $P_0$. The other one then provides a path from $P_1$ to $P_2$ in $\mathbb{R}^2 \setminus P_0$. □

Note that both of these proofs get you full points: you are safe to assume basic facts from topology and anything we discussed/read about differential topology unless stated otherwise.

**3. Compute the tangent space of the ellipsoid $x^2 + y^2 + 2z^2 = 1$ at the point $P = (0, 0, \sqrt{2}/2)$.**
**First proof.** Let $X \subset \mathbb{R}^3$ be the ellipsoid as above and $P = (0, 0, \sqrt{2}/2)$. Consider the map $f : X \to \mathbb{R}^2$ defined by $f : (x, y, z) \mapsto (x, y)$: note it is smooth, as it is the restriction of the standard projectio function $\pi_{x,y} : \mathbb{R}^3 \to \mathbb{R}^2$. Let us define a local inverse to $f$ near $P$, given by $\psi : U \to X$ be the map $(x, y) \mapsto (x, y, \sqrt{\frac{1-(x^2+y^2)}{2}})$, on $U = \{(x, y) \mid x^2 + y^2 < 1$ (strict inequality implies $U$ is the continuous preimage of an open ray under the function $x^2+y^2$, hence open). Since we are considering the square root of a *positive* smooth function, the third coordinate of $g$, hence $g$ itself, is smooth on $U$. Let $N_P$ be defined as $\psi(U)$. Then $N_P$ is the intersection of $X$ with the open $\{(x, y, z) \mid z \geq 0\} \subset \mathbb{R}^3$, hence open, and $\psi : U \to N_P$ is smooth as we have seen. The function $\phi = f \mid_{N_P}$ is smooth as it is the restriction of a smooth function. Evidently, $\phi : U \to N_P$ and $\psi : N_P \to U$ are inverse functions, hence $\psi$ is a chart. We compute $\partial_x\psi(0, 0) = (1, 0, 0)$ and $\partial_y\psi = (0, 1, 0)$, so the tangent space is $\langle \hat{x}, \hat{y} \rangle \subset \mathbb{R}^3$.

**Alternative proof.** Let $X \subset \mathbb{R}^3$ be the ellipsoid above and $P = (0, 0, \sqrt{2}/2)$. Consider the function $F : \mathbb{R}^3 \to \mathbb{R}$ given by

$$F : (x, y, z) \mapsto x^2 + y^2 + 2z^2.$$

Then $X = F^{-1}(1)$. We compute the differential (equivalently, Jacobian) of $F$ at $P$, obtaining the $1 \times 3$ matrix $d_P(F) = (0, 0, \sqrt{2})$. The corresponding linear transformation $\mathbb{R}^3 \to \mathbb{R}$ is nonzero hence surjective, so $F$ is a submersion at $P$ and by the local submersion theorem we can compute the tangent to $F^{-1}(1)$ by taking the kernel,

$$\text{Ker}\,(d_P(F)) = \{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \mid x, y \in \mathbb{R}\}.$$

**4. (a) Let $X \subset \mathbb{R}^N$ be a manifold of dimension $n \geq 1$ and $P \in X$ a point. Show that there is a coordinate $1 \leq i \leq N$ such that the $i$th coordinate map $\pi_i : X \to \mathbb{R}$ defined by $\pi_i(x_1, \ldots, x_N) := x_i$ is a submersion at $P$.**
$T_P(X)$ is an $n$-dimensional linear subspace in $\mathbb{R}^N$. Since $n \geq 1$ there is a nonzero tangent vector $v = (x_1, \ldots, x_N)^T$, with $x_i \neq 0$. Now as the projection function $\pi_i : (x_1, \ldots, x_n) \mapsto x_i$ is linear, we have $d_Q\pi_i = \pi_i$ for all $Q \in \mathbb{R}^N$, and so $d\pi_i(v) = \pi_i(v) = x_i \neq 0$.
**(b) Show more generally that it is possible to choose $n$ different coordinate maps which are independent at $P$, i.e. such that the resulting map $X \to \mathbb{R}^n$ is a local diffeomorphism at $P$.**
Similar to the above. Let $\{v_1, \ldots, v_n\}$ be a basis for $T_PX$, with each $v_i \in \mathbb{R}^N$. Let $M = (v_1| \ldots |v_n)$ be this basis written as a matrix. This matrix has rank $n$,

so it has a nonvanishing $n \times n$ minor. Let $(i_1, \dots, i_n)$ be the indices of the rows of this minor. Then the projection function $\pi : (x_1, \dots, x_N) \mapsto (x_{i_1}, dots, x_{i_n})$ is linear, thus $d_P\pi = \pi$, and therefore $d_P\pi : T_PX \to \mathbb{R}^n$ is a bijection.

Alternative proof sketch: find the first $\pi_{i_1}$ using $(a)$. Set $\pi_{i_1}(P) = x$, then use the preimage theorem to show that $X' := \pi_{i_1}^{-1}(x) \cap N_P$ is a submanifold (for some neighborhood $N_P$), and then inductively apply (a) to $X'$ to get coordinates $i_2, \dots, i_n$.

**(c) Deduce that for any $X \in \mathbb{R}^3$ a two-dimensional surface and any $P \in X$, there is a chart $\psi : U \to X$ for a neighborhood of $P$ with $U \subset \mathbb{R}^2$ and such that either $\psi(x, y) = (x, y, f(x, y))$ or $\psi(x, y) = (x, f(x, y), y)$ or $\psi(x, y) = (f(x, y), x, y)$ for some function $f : \mathbb{R}^2 \to \mathbb{R}$.**

By part (b), there is a coordinate projection $\pi_{ij} : \mathbb{R}^3 \to \mathbb{R}^2$ which is a local differomorphism on $X$ near $P$. There are three options for $\pi_{ij}$ (depending on which coordinate gets projected *along*), namely $\pi_{12}, \pi_{23}, \pi_{13}$.

**Case I: assume** $\pi_{12}$ is a local diffeomorphism. Then locally $\pi_{12} : X \to \mathbb{R}^2$ has an inverse, i.e. there is a neighborhood $N_P$ in $X$ of $P$ such that $\pi_{12} : N_P \to \pi_{12}(N_P)$ is a diffeomorphism. Since $\pi_{12}$ is a local diffeomorphism on $N_P$ (in particular, a submersion), we know that the image of an open is open, so $\pi_{12}(N_P)$ is open. Call it $U$. Then $\psi := \big(\pi_{12} \mid_{N_P}\big)^{-1} : U \to N_P$ is the inverse diffeomorphism. Now if $\psi(x, y) = \big(\psi_1(x, y), \psi_2(x, y), \psi_3(x, y)\big)$ then using $\pi_{12}\psi(x, y) = (x, y)$ we deduce that $\psi_1(x) = x$ and $\psi_2(y) = y$. Setting $f(x, y) = \psi_3(x, y)$ we are done in this case.

**Other cases:** if $\pi_{23}$ is a diffeomorphism then by a similar argument we construct $\psi : U \to N_P$ with $\psi(x, y) = (x, f(x, y), y)$ and if $\pi_{13}$ is a diffeomorphism then by a similar argument we construct $\psi : U \to N_P$ with $\psi(x, y) = \big(f(x, y), x, y\big)$. □

**5.** Show that the map $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $t \mapsto (\cos(t), \sin(t))$ is an immersion.
Take its differential: $d_t\alpha = (-\sin(t), \cos(t))^T$, which has magnitude 1, hence non-zero (and therefore the corresponding map $\mathbb{R} \to \mathbb{R}^2$ is injective).

**6. Let $I = (0, 1)$ be the unit interval. Give an example of a map from the disjoint union $(0, 1) \sqcup (1, 2)$ to $\mathbb{R}^2$ which is an immersion and injective but not an embedding.**
Consider the "letter $T$" map,

$$x \mapsto \begin{cases} (x - 1/2, 0) & x \in (0, 1) \\ (0, 1 - x) & x \in (1, 2) \end{cases}.$$

Draw this map: its image looks like the capital letter $T$. The map is locally linear near each point, so clearly an immersion. It is injective, as no point on the second interval has image with $y$-coordinate zero. But it does not have smooth inverse, as the sequence of points $f(1 + 1/n)$ converges to $f(1/2)$.

**7. (a) Prove that the map $sq : GL_n \to GL_n$ given by $M \mapsto M^2$ is a local diffeomorphism at $I$ (hint: compute its differential).** To compute the differential, let $M \in \mathrm{Mat}_n = T_I GL_n$. Then $d_I sq(v) = \lim_{\epsilon \to 0} \frac{(I+\epsilon M)^2 - I}{\epsilon} = \lim_\epsilon \frac{2\epsilon M + \epsilon^2 M^2}{\epsilon} = \lim_\epsilon 2M + \epsilon M^2 = 2M$. Thus the differential is given by the linear map $M \to 2M$.

**(b) Is this map a diffeomorphism? (Hint: look at the determinant.)** No: it is not surjective, since $|M^2| = |M|^2 > 0$, so $sq(M)$ has positive determinant and a matrix such as $\begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ with negative determinant cannot be in the image.

**(c) Deduce that for any Lie group $G$, the map $sq : G \to G$ is a local diffeomorphism.** Since $G$ is a Lie group, for any $M \in G$ we have $M^2 \in G$ as well, so the map $sq$ can be restricted to $G$. By the chain rule, its differential will then coincide with the restriction to $T_I G$ of $d_I sq$. Thus it will be the multiplication-by-two map on the linear space $T_I G$. This is an invertible (indeed, a scalar) matrix, therefore $sq_G$ is a local diffeomorphism.

**8. True or false/short answer**:

**(a) Is the sphere $S^2$ connected? Is it compact?** Yes to both. Any two points can be connected by a path along the equator, and it is closed and bounded in $\mathbb{R}^3$, hence compact.

**(b) If $f : X \to Y$ takes $P$ to $Q$, there is a pair of parametrizations $\psi_P : U \to X, \psi_Q : V \to Y$ such that $U \subset T_P(X), V \subset T_Q(Y)$ are open and the composed map $U \to V$ is given by $d_P f$.** This is true in the case that $f$ is either a submersion or an immersion (equivalently: maximal rank). But not in general: for example the map $sq : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^2$ has differential 0 at $x = 0$, but cannot be expressed as the zero map for any smooth parametrization of any neighborhood $0 \in \mathbb{R}$ (as this would imply that there is an open neighborhood of 0 of points that all square to zero, which is absurd).

**(c) A map is a local diffeomorphism if and only if it is both an immersion and a submersion** True, since a linear map is invertible iff it is injective and surjective.

**(d) The preimage of a critical value of a function $f : X \to Y$ is smooth.** False: this is true (by the preimage theorem) for a *regular* value, but not for a critical value. For example consider the preimage $F^{-1}(0)$ of the function $F : \mathbb{R}^2 \to \mathbb{R}$ given by $F(x, y) = xy$ at the critical value $0 \in \mathbb{R}$.

**(e) If a function $f : X \to Y$ is a submersion then every $Q \in Y$ is a regular value.** True: a value $Q \in Y$ is regular if and only if $f$ is a submersion locally for each $P \in f^{-1}(Q)$, and if $f$ is a submersion then it is a submersion locally near every $P$. (In fact, the converse is also true: if every $Q \in Y$ is a regular value then $f$ is a submersion.)