# Math 141 Midterm 1 and Final exam practice
# Topology, immersions, submersions, Lie groups.

**1.** Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous. Show that its graph $\Gamma(f) \subset \mathbb{R}^{m+n}$, defined by $\Gamma(f) := \{(x, y) \mid f(x) = y\} \subset \mathbb{R}^m \times \mathbb{R}^n$ is closed.

**2.** Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

**3.** Compute the tangent space of the ellipsoid $x^2 + y^2 + 2z^2 = 1$ at the point $P = (0, 0, \sqrt{2}/2)$.

**4.** (a) Let $X \subset \mathbb{R}^N$ be a manifold of dimension $n \geq 1$ and $P \in X$ a point. Show that there is a coordinate $1 \leq i \leq N$ such that the $i$th coordinate map $\pi_i : X \to \mathbb{R}$ defined by $\pi_i(x_1, \ldots, x_N) := x_i$ is a submersion at $P$.

(b) Show more generally that it is possible to choose $n$ different coordinate maps which are independent at $P$, i.e. such that the resulting map $X \to \mathbb{R}^n$ is a local diffeomorphism at $P$.

(c) Deduce that for any $X \in \mathbb{R}^3$ a two-dimensional surface and any $P \in X$, there is a chart $\psi : U \to X$ for a neighborhood of $P$ with $U \subset \mathbb{R}^2$ and such that either $\psi(x, y) = (x, y, f(x, y))$ or $\psi(x, y) = (x, f(x, y), y)$ or $\psi(x, y) = (f(x, y), x, y)$ for some function $f : \mathbb{R}^2 \to \mathbb{R}$.

**5.** Show that the map $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $t \mapsto (\cos(t), \sin(t))$ is an immersion.

**6.** Let $I = (0, 1)$ be the unit interval. Give an example of a map from the disjoint union $(0, 1) \sqcup (1, 2)$ to $\mathbb{R}^2$ which is an immersion and injective but not an embedding.

**7.** (a) Prove that the map $sq : GL_n \to GL_n$ given by $M \mapsto M^2$ is a local diffeomorphism at $I$ (hint: compute its differential).

(b) Is this map a diffeomorphism? (Hint: look at the determinant.)

(c) Deduce that for any Lie group $G$, the map $sq : G \to G$ is a local diffeomorphism.

**8.** True or false/short answer:

(a) Is the sphere $S^2$ connected? Is it compact?

(b) If $f : X \to Y$ takes $P$ to $Q$, there is a pair of parametrizations $\psi_P : U \to X, \psi_Q : V \to Y$ such that $U \subset T_P(X), V \subset T_Q(Y)$ are open and the composed map $U \to V$ is given by $d_P f$.

(c) A map is a local diffeomorphism if and only if it is both an immersion and a submersion.

(d) The preimage of a critical value of a function $f : X \to Y$ is smooth.

(e) If a function $f : X \to Y$ is a submersion then every $Q \in Y$ is a regular value.