map $\phi \times \psi : W \times U \longrightarrow X \times Y$ by the formula

$$\phi \times \psi(w, u) = (\phi(w), \psi(u)).$$

Of course, $W \times U$ is an open set in $\mathbf{R}^k \times R^l = \mathbf{R}^{k+l}$, and it is easy to check that $\phi \times \psi$ is a local parametrization of $X \times Y$ around (x, y). (Check this point carefully, especially verifying that $(\phi \times \psi)^{-1}$ is a smooth map on the not-necessarily-open set $X \times Y \subset \mathbf{R}^{M+N}$). Since this map is a local parametrization around an arbitrary point $(x, y) \in X \times Y$, we have proved:

Theorem. If X and Y are manifolds, so is $X \times Y$, and $\dim X \times Y = \dim X + \dim Y$.

We mention another useful term here. If X and Z are both manifolds in $\mathbf{R}^N$ and $Z \subset X$, then Z is a *submanifold* of X. In particular, X is itself a submanifold of $\mathbf{R}^N$. Any open set of X is a submanifold of X.

The reader should be warned that the slothful authors will often omit the adjective "smooth" when speaking of mappings; nevertheless, smoothness is virtually always intended.

EXERCISES

1. If $k < l$ we can consider $\mathbf{R}^k$ to be the subset $\{(a_1, \ldots, a_k, 0, \ldots, 0)\}$ in $\mathbf{R}^l$. Show that smooth functions on $\mathbf{R}^k$, considered as a subset of $\mathbf{R}^l$, are the same as usual.

***2.** Suppose that X is a subset of $\mathbf{R}^N$ and Z is a subset of X. Show that the restriction to Z of any smooth map on X is a smooth map on Z.

***3.** Let $X \subset \mathbf{R}^N$, $Y \subset \mathbf{R}^M$, $Z \subset \mathbf{R}^L$ be arbitrary subsets, and let $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ be smooth maps. Then the composite $g \circ f : X \longrightarrow Z$ is smooth. If f and g are diffeomorphisms, so is $g \circ f$.

4. (a) Let B_a be the open ball $\{x : |x|^2 < a\}$ in $\mathbf{R}^k$. ($|x|^2 = \sum x_i^2$) Show that the map

$$x \longrightarrow \frac{ax}{\sqrt{a^2 - |x|^2}}$$

is a diffeomorphism of B_a onto $\mathbf{R}^k$. [HINT: Compute its inverse directly.]

(b) Suppose that X is a k-dimensional manifold. Show that every point in X has a neighborhood diffeomorphic to all of $\mathbf{R}^k$. Thus local parametrizations may always be chosen with all of $\mathbf{R}^k$ for their domains.

*5. Show that every k-dimensional vector subspace V of $\mathbf{R}^N$ is a manifold diffeomorphic to $\mathbf{R}^k$, and that all linear maps on V are smooth. If $\phi: \mathbf{R}^k \to V$ is a linear isomorphism, then the corresponding coordinate functions are linear functionals on V, called *linear coordinates.*

6. A smooth bijective map of manifolds need not be a diffeomorphism. In fact, show that $f: \mathbf{R}^1 \to \mathbf{R}^1, f(x) = x^3$, is an example.

7. Prove that the union of the two coordinate axes in $\mathbf{R}^2$ is not a manifold. (HINT: What happens to a neighborhood of 0 when 0 is removed?)

8. Prove that the paraboloid in $\mathbf{R}^3$, defined by $x^2 + y^2 - z^2 = a$, is a manifold if $a > 0$. Why doesn't $x^2 + y^2 - z^2 = 0$ define a manifold?

9. Explicitly exhibit enough parametrizations to cover $S^1 \times S^1 \subset \mathbf{R}^4$.

10. "The" *torus* is the set of points in $\mathbf{R}^3$ at distance b from the circle of radius a in the xy plane, where $0 < b < a$. Prove that these tori are all diffeomorphic to $S^1 \times S^1$. Also draw the cases $b = a$ and $b > a$; why are these not manifolds?

11. Show that one cannot parametrize the k sphere S^k by a single parametrization. [HINT: S^k is compact.]

*12. Stereographic projection is a map π from the punctured sphere $S^2 - \{N\}$ onto $\mathbf{R}^2$, where N is the north pole (0, 0, 1). For any $p \in S^2 - \{N\}$, $\pi(p)$ is defined to be the point at which the line through N and p intersects the xy plane (Figure 1-6). Prove that $\pi : S^2 - \{N\} \to \mathbf{R}^2$ is a diffeomorphism. (To do so, write π explicitly in coordinates and solve for π^{-1}.)

Note that if p is near N, then $|\pi(p)|$ is large. Thus π allows us to think of S^2 a copy of $\mathbf{R}^2$ compactified by the addition of one point "at infinity." Since we can define stereographic projection by using the

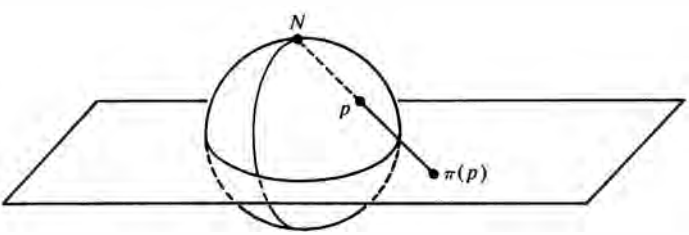

Figure 1-6

south pole instead of the north, S^2 may be covered by two local parametrizations.

***13.** By generalizing stereographic projection define a diffeomorphism $S^k - \{N\} \longrightarrow \mathbf{R}^k$.

***14.** If $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ are smooth maps, define a *product map* $f \times g : X \times Y \longrightarrow X' \times Y'$ by

$$(f \times g)(x, y) = (f(x), g(y)).$$

Show that $f \times g$ is smooth.

15. Show that the projection map $X \times Y \longrightarrow X$, carrying (x, y) to x, is smooth.

***16.** The *diagonal* Δ in $X \times X$ is the set of points of the form (x, x). Show that Δ is diffeomorphic to X, so Δ is a manifold if X is.

***17.** The *graph* of a map $f: X \longrightarrow Y$ is the subset of $X \times Y$ defined by

$$\text{graph}\,(f) = \{(x, f(x)) : x \in X\}.$$

Define $F: X \longrightarrow \text{graph}\,(f)$ by $F(x) = (x, f(x))$. Show that if f is smooth, F is a diffeomorphism; thus graph (f) is a manifold if X is. (Note that $\Delta = \text{graph (identity)}$.)

***18.** (a) An extremely useful function $f: \mathbf{R}^1 \longrightarrow \mathbf{R}^1$ is

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Prove that f is smooth.

(b) Show that $g(x) = f(x - a)g(b - x)$ is a smooth function, positive on (a, b) and zero elsewhere. (Here $a < b$.) Then

$$h(x) = \frac{\int_{-\infty}^{x} g\,dx}{\int_{-\infty}^{\infty} g\,dx}$$

is a smooth function satisfying $h(x) = 0$ for $x < a$, $h(x) = 1$ for $x > b$, and $0 < h(x) < 1$ for $x \in (a, b)$.

(c) Now construct a smooth function on $\mathbf{R}^k$ that equals 1 on the ball of radius a, zero outside the ball of radius b, and is strictly between 0 and 1 at intermediate points. (Here $0 < a < b$.)

pend on parametrization. Do write down the calculation yourself to be sure that you understand.

Does our extension of the derivative to maps of arbitrary manifolds truly conform to the chain rule as we intended? (It had better, for we showed that it is the only possibility.) Let $g\colon Y \longrightarrow Z$ be another smooth map, and let $\eta : W \longrightarrow Z$ parametrize Z about $z = g(y)$. Here $W \subset \mathbf{R}^m$ and $\eta(0) = z$. Then from the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\;f\;} & Y & \xrightarrow{\;g\;} & Z \\ \uparrow{\scriptstyle \phi} & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \eta} \\ U & \xrightarrow[h = \psi^{-1} \circ f \circ \phi]{} & V & \xrightarrow[j = \eta^{-1} \circ g \circ \psi]{} & W \end{array}$$

we derive the square

$$\begin{array}{ccc} X & \xrightarrow{\;g \circ f\;} & Z \\ \uparrow{\scriptstyle \phi} & & \uparrow{\scriptstyle \eta} \\ U & \xrightarrow[j \circ h]{} & W \end{array}$$

Thus, by definition,

$$d(g \circ f)_x = d\eta_0 \circ d(j \circ h)_0 \circ d\phi_0^{-1}.$$

By the chain rule for maps of open subsets of Euclidean spaces, $d(j \circ h)_0 = (dj)_0 \circ (dh)_0$. Thus

$$d(g \circ f)_x = (d\eta_0 \circ dj_0 \circ d\psi_0^{-1}) \circ (d\psi_0 \circ dh_0 \circ d\phi_0^{-1}) = dg_y \circ df_x.$$

We have proved the

Chain Rule. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are smooth maps of manifolds, then

$$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$

EXERCISES

***1.** For a submanifold X of Y, let $i\colon X \longrightarrow Y$ be the inclusion map. Check that di_x is the inclusion map of $T_x(X)$ into $T_x(Y)$.

***2.** If U is an open subset of the manifold X, check that

$$T_x(U) = T_x(X) \quad \text{for} \quad x \in U.$$

3. Let V be a vector subspace of $\mathbf{R}^N$. Show that $T_x(V) = V$ if $x \in V$.

*4. Suppose that $f: X \longrightarrow Y$ is a diffeomorphism, and prove that at each x its derivative df_x is an isomorphism of tangent spaces.

5. Prove that $\mathbf{R}^k$ and $\mathbf{R}^l$ are not diffeomorphic if $k \neq l$.

6. The tangent space to S^1 at a point (a, b) is a one-dimensional subspace of $\mathbf{R}^2$. Explicitly calculate the subspace in terms of a and b. [The answer is obviously the space spanned by $(-b, a)$, but prove it.]

7. Similarly exhibit a basis for $T_p(S^2)$ at an arbitrary point $p = (a, b, c)$.

8. What is the tangent space to the paraboloid defined by $x^2 + y^2 - z^2 = a$ at $(\sqrt{a}, 0, 0)$, where $(a > 0)$?

*9. (a) Show that for any manifolds X and Y,

$$T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y).$$

(b) Let $f: X \times Y \longrightarrow X$ be the projection map $(x, y) \longrightarrow x$. Show that

$$df_{(x,y)} : T_x(X) \times T_y(Y) \longrightarrow T_x(X)$$

is the analogous projection $(v, w) \longrightarrow v$.

(c) Fixing any $y \in Y$ gives an injection mapping $f: X \longrightarrow X \times Y$ by $f(x) = (x, y)$. Show that $df_x(v) = (v, 0)$.

(d) Let $f: X \longrightarrow X'$, $g: Y \longrightarrow Y'$ be any smooth maps. Prove that

$$d(f \times g)_{(x,y)} = df_x \times dg_y.$$

*10. (a) Let $f: X \longrightarrow X \times X$ be the mapping $f(x) = (x, x)$. Check that $df_x(v) = (v, v)$.

(b) If Δ is the diagonal of $X \times X$, show that its tangent space $T_{(x,x)}(\Delta)$ is the diagonal of $T_x(X) \times T_x(X)$.

*11. (a) Suppose that $f: X \longrightarrow Y$ is a smooth map, and let $F: X \longrightarrow X \times Y$ be $F(x) = (x, f(x))$. Show that

$$dF_x(v) = (v, df_x(v)).$$

(b) Prove that the tangent space to graph (f) at the point $(x, f(x))$ is the graph of $df_x: T_x(X) \longrightarrow T_{f(x)}(Y)$.

*12. A *curve* in a manifold X is a smooth map $t \longrightarrow c(t)$ of an interval of $\mathbf{R}^1$ into X. The *velocity vector* of the curve c at time t_0—denoted simply

$dc/dt(t_0)$—is defined to be the vector $dc_{t_0}(1) \in T_{x_0}(X)$, where $x_0 = c(t_0)$ and $dc_{t_0}\colon \mathbf{R}^1 \longrightarrow T_{x_0}(X)$. In case $X = \mathbf{R}^k$ and $c(t) = (c_1(t), \ldots, c_k(t))$ in coordinates, check that

$$\frac{dc}{dt}(t_0) = (c_1'(t_0), \ldots, c_k'(t_0)).$$

Prove that every vector in $T_x(X)$ is the velocity vector of some curve in X, and conversely. [HINT: It's easy if $X = \mathbf{R}^k$. Now parametrize.]

§3 The Inverse Function Theorem and Immersions

Before we really begin to discuss the topology of manifolds, we must study the local behavior of smooth maps. Perhaps the best reason for always working with smooth maps (rather than continuous maps, as in non-differential topology) is that local behavior is often entirely specified, up to diffeomorphism, by the derivative. The elucidation of this remark is the primary objective of the first chapter.

If X and Y are smooth manifolds of the same dimension, then the simplest behavior a smooth map $f\colon X \longrightarrow Y$ can possibly exhibit around a point x is to carry a neighborhood of x diffeomorphically onto a neighborhood of $y = f(x)$. In such an instance, we call f a *local diffeomorphism* at x. A necessary condition for f to be a local diffeomorphism at x is that its derivative mapping $df_x : T_x(X) \longrightarrow T_y(Y)$ be an isomorphism. (See Exercise 4 in Section 2). The fact that this linear condition is also sufficient is the key to understanding the remark above.

The Inverse Function Theorem. Suppose that $f\colon X \longrightarrow Y$ is a smooth map whose derivative df_x at the point x is an isomorphism. Then f is a local diffeomorphism at x.

The Inverse Function Theorem is a truly remarkable and valuable fact. The derivative df_x is simply a single linear map, which we may represent by a matrix of numbers. This linear map is nonsingular precisely when the determinant of its matrix is nonzero. Thus the Inverse Function Theorem tells us that the seemingly quite subtle question of whether f maps a neighborhood of x diffeomorphically onto a neighborhood of y reduces to a trivial matter of checking if a single number—the determinant of df_x—is nonzero!

You have probably seen a proof of the Inverse Function Theorem for the special case when X and Y are open subsets of Euclidean space. One may be found in any text on calculus of several variables—for example, Spivak [2]. You should easily be able to translate the Euclidean result to the manifold setting by using local parametrizations.