Math 141 final practice solutions.

2.2.5 Let $f : [-1,1] \to [-1,1]$ be a map. Consider $g(x) := f(x) - x$. Then $f(0) \geq -1$ and $f(1) \leq 1$ (as both are in $[-1,1]$) so $g(-1) \geq 0, g(1) \leq 0$, and apply the intermediate value theorem.

2.2.7 (on the hard side.) Using the hint, WLOG f has no kernel and is therefore invertible. Define a function $g : S^{n-1} \to S^{n-1}$ by $g(v) := \frac{f(v)}{|f(v)|}$. Note that this is a smooth function, as f is smooth and $v \mapsto |v|$ is smooth outside of zero, therefore by invertibility of f, g is smooth on the image $f(S^{n-1})$.

Let $Q = [0,\infty)^n \cap S^{n-1}$ be the positive quadrant of the sphere. Then for $v \in Q$ we see that the vector $f(v)$ has all nonnegative coefficients (by positivity of coefficients of f), so $g(v) = \frac{f(v)}{|v|}$ is in Q. Now Q is diffeomorphic to the disk D^{n-1} (if you want to prove this, first note that Q_n is diffeomorphic (via projection from the origin) to the "simplex" $\Delta^{n-1} := \{(x_1, \ldots, x_n) \mid x_i \geq 0, \sum x_i = 1$, which is the intersection of the upper quadrant with a skew hyperplane. Then use that the shifted set $\Delta^{n-1} - (1/n, 1/n, \ldots, 1/n)$ is a convex set in the hyperplane defined by $\sum x_i = 0$, and any compact convex set $C \subset V$ of a vector space is homeomorphic to the disk by the map $v \mapsto \frac{v}{r_v}$ for r_v the length of the radius of C in the direction of v).

Now apply the previous problem, 7, to deduce that the map $g : Q \to Q$ has a fixed point, $v\ inQ$, satisfying $\frac{f(v)}{|f(v)|} = v$. Thus v is an eigenvector with positive eigenvalue $|v|$.

2.3.5 Let $f = i_X : X \to Y$ be the inclusion. First: for M the dimension of the ambient space of Y and $\mathring{B}^M$ the open unit M-disk, we have constructed a map $F : X \times \mathring{B}^M \to Y$ which is a submersion (hence transversal to Z) with $f(p) = F(p, 0)$. By compactness of X, there is a δ_1 such that $v < \delta_1$ implies f_v is at most ϵ away from X. Stability of embeddings (combined with the result of 1.6.11) implies that there is a δ_2 with f_v an embedding for $|v| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$. By the transversality theorem, the set of $v \in B^M$ for which $f_v \pitchfork Z$ is dense, thus it contains points in the open ball $\mathring{B}^M_\delta \subset \mathring{B}^M$. For v such a point, first $|v| < \delta_1$ implies f_v is distance at most ϵ from f, secondly $|v| < \delta_2$ implies f_v is an embedding and by assumption on v, we have $f_v \pitchfork Z$.

(Alternative to denseness: consider the function $F_{<\delta} : F \mid X \times \mathring{B}^M_\delta$ (restriction to the open δ-disk). Then G is the restriction of a submersion to an open subset, hence still a submersion, so there is some $v \in \mathring{B}^M_\delta$ which satisfies transversality).

Alternative proof using a fact mentioned in class: I said that for any $f : X \to Y$ there is a homotopy f_t with $f_0 = f$ such that outside of a measure zero subset of $t \in [0,1]$, we have $f_t \pitchfork Z$; in particular, such transversal t are dense. Now for a compact manifold, both the property of being an immersion and the property of being distance $< \epsilon$ from f are stable. Hence for some $\delta > 0$, we have f_t an immersion of distance $< \epsilon$ from f. By denseness, there is a value $t < \delta$ for which $f_t \pitchfork Z$.

2.3.7 This problem is actually false as stated. Example: X is the single point in the origin (or X is a circle through the origin in $\mathbb{R}^3$ and the dimension $l = 1$). **It is true if we assume in addition that X does not pass through the origin.** Assume X does not pass through the origin[1]. Consider the map $\alpha : GL_N \times \mathbb{R}^k \to \mathbb{R}^N$, given by $\alpha(A, v) = A \cdot v$ (action of an invertible matrix A on a vector). Then for any fixed $A \in GL_N$, the map α_A is an embedding of V in $\mathbb{R}^N$. Now the tangent space $T_A GL_N = Mat_{N \times N}$ is the N^2-dimenisional space of matrices, and for $M \in T_A GL_N$, we compute

$$T_{v,A}(M) = \lim_{\epsilon \to 0} \frac{(A + \epsilon M)v - Av}{\epsilon} = Mv.$$

Note that if $v \in \mathbb{R}^N$ is nonzero, the set of vectors $\{Mv \mid M \in \mathrm{Mat}_{N \times N}\}$ is all of $\mathbb{R}^N$, hence $T_{A,v} : \mathrm{Mat}_{N \times N} \times \mathbb{R}^k \to \mathbb{R}^N$ is surjective (already so on the first component $\mathrm{Mat}_{N \times N}$ of the tangent space). Now if $w \in \mathbb{R}^n$ is a nonzero vector and $\alpha(A, v) = Av = w$ then v is a nonzero vector and thus α is a submersion at $p = (A, v)$. Since by assumption, X does not pass through zero, this implies that all points of X are regular values for α, so $\alpha \pitchfork X$ for each point of X. Therefore, for some choice of A, the map $\alpha_A : \mathbb{R}^l \to \mathbb{R}^N$ is transversal to X. As α_A is an embedding, this is equivalent to transversality of X with the hyperplane $A(\mathbb{R}^k)$.

2.4.4 We are given that there is a homotopy from f to a *constant* map f_1, i.e. such that $f_1(p) = q_1 \forall p \in X$, for some fixed $q_0 \in Y$. We are given that X is at least 1-dimensional. Then $\dim(X) + \dim(Z) = \dim(Y)$ implies Z is at most $\dim(Y) - 1$-dimensional, so the image of the inclusion i_Z has measure zero, and a dense set of points of Y is not in Z. Choose a point q_2 in $Y \setminus Z$ in the same connected component as q_1 (possible for example by denseness, alternatively by replacing Y by a single connected component). Then any path from q_1 to q_2 defines a homotopy from f_1 to the constant map f_2 which maps all of x to the point q_2. Now by assumption, q_2 is disjoint from Z, thus vacuously transversal; its (mod 2) intersection number is zero. By transversality of homotopy, this implies that $I_2(f_1, Z) = 0$, hence $I_2(f_0, Z) = 0$.

2.4.7 Look at the identity map $\mathrm{id} : S^1 \to S^1$. This map is a diffeomorphism, hence a submersion, hence transversal to a single-point manifold $p \in S^1$. The mod 2 intersection number $I_2(\mathrm{id}, \{p\})$, equal to the mod 2 degree of id, is equal to 1. On the other hand, if S^1 were simply connected, this would imply that any map $S^1 \to S^1$, including the identity map, is homotopic to a constant map, which would imply by 2.4.4 above that (as S^1 is ≥ 1-dimensional,) $I_2(\mathrm{id}, p) = 0$, contradiction.

2.4.12 This follows from the fact that a map from a compact to a noncompact topological space cannot be surjective (the image of a compact set under a

[1] With a little bit of extra work, you can eliminate the condition that X does not pass through zero if the sum of dimensions $l + n \geq N$.

continuous map is compact; this is easiest to understand via our other definition of compactness: the image of a closed bounded subset under a continuous map is closed and bounded). This means that there is some point $q \in Y$ with no preimage. This point is vacuously a regular value, so

$$\deg_2(f) = |f^{-1}(q)| \mod 2 = 0.$$

2.6.1 The Borsuk-Ulam theorem states that if $f : S^{n-1} \to \mathbb{R}^n \setminus 0$ carries antipodal points to antipodal points, then $W_2(f) = 1$. Here $W_2(f) := I_2(f, R)$ for $R = \{(0, \dots, 0, r)\}$ a ray. Let $F : \mathbb{R}^n \setminus 0 \to S^n$ be the unit vector map, $v \mapsto \frac{v}{|v|}$ (continuous outside of $v = 0$ as we have seen above). Then $R = F^{-1} e_n$ for $e_n = (0, \dots, 0, 1)$. Note that $F^{-1}(e_n) = \{(0, \dots, 0, r) \mid r > 0\} \subset \mathbb{R}^n$. We know that $T_{e_n} S^1 = e_n^{\perp} = \{(x_1, \dots, x_{n-1}, 0) \mid x_i \in \mathbb{R}\}$, the standard $n-1$-dimensional hyperplane. For $p = (0, \dots, 0, r)$ with $r > 0$, the differential $d_p(F)$ can be computed out to

$$\begin{pmatrix} 1/r & 0 & \dots & 0 \\ 0 & 1/r & \dots & 0 \\ \dots & \ddots & \dots & 0 \\ 0 & 0 & \dots & 1/r \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

This is surjective, so e_n is a regular value of F. Thus by our result about transversality and regular values, $f \pitchfork R$ iff e_n is a regular value of the composed map $F \circ f : S^{n-1} \to S^{n-1}$, and since $f^{-1}(R) = (F \circ f)^{-1}(e_n)$ (and this remains true if f is replaced by a homotopic map), we see that $I_2(f, R) = I_2(F \circ f, \{e_n\})$. Intersection number with a single point is equal to the degree, so $1 = I_2(f, R) = \deg_2(F \circ f)$. Now if $f : S^{n-1} \to S^{n-1}$ already has image in $S^{n-1} \subset \mathbb{R}^n$, define $\tilde{f} = i_{S^{n-1}} \circ f : S^{n-1} \to \mathbb{R}^n$ (considered as a map $S^{n-1} \to \mathbb{R}^n$). The above argument then gives that $\deg(\tilde{f} \circ F) = 1$, but in this case $\tilde{f} \circ F = f : S^{n-1} \to S^{n-1}$, completing the proof.