Math 143 Elementary Algebraic Geometry, Fall 2018. Instructor: Dmitry Tonkonog

Practice Midterm

The actual midterm will be 80 minutes long. No notes, books or calculators are allowed.

Problem 1. Consider $\mathbb{C}^3$ with coordinates $x, y, z \in \mathbb{C}$. Let $V \subset \mathbb{C}^3$ be the union of the plane $\{z = 0\}$ with the line $\{x = y = 0\}$.

(a) Show that V is an affine variety.

(b) Let $I \subset \mathbb{C}[x, y, z]$ be the ideal consisting of all polynomials $f(x, y, z)$ such that $f(p) = 0$ for all $p \in V$. Find a finite set of polynomials generating I (with a proof).

Problem 2.

(a) What does it mean for an ideal $I \subset A$ to be *maximal*? Write the definition.

(b) Prove that every maximal ideal in $\mathbb{C}[x]$ has the form $(x - a)$ for some $a \in \mathbb{C}$.

Problem 3. Consider $\mathbb{C}P^2$ with homogeneous coordinates $[x : y : z]$. Consider the following 5 points in $\mathbb{C}P^2$:

$p_1 = [0 : 0 : 1]$,

$p_2 = [1 : 0 : 1]$,

$p_3 = [2 : 0 : 1]$,

$p_4 = [0 : 1 : 1]$,

$p_5 = [0 : 2 : 1]$.

Consider 5 other points $q_1, \ldots, q_5 \in \mathbb{C}P^2$ which are pairwise distinct and all belong to the set $\{x^2 + y^2 + z^2 = 0\} \subset \mathbb{C}P^2$.

Does there exist a projective transformation $\phi\colon \mathbb{C}P^2 \to \mathbb{C}P^2$ such that $\phi(p_i) = q_i$, $i = 1, \ldots, 5$?

Hint: find a conic passing through $p_1, \ldots, p_5$. Is it smooth?

Problem 4. Consider the following system of equations in $\mathbb{C}^2$:

$$\begin{cases} y = x^2 \\ y = x^3. \end{cases}$$

(a) Rewrite it in projective coordinates $[x : y : z]$ for $\mathbb{C}P^2$.

(b) Find all solutions to this system of equations in $\mathbb{C}P^2$.

Problem 5. (a) Consider the ideal generated by $I = (x - 1)$ in the ring $A = \mathbb{C}[x, y]/(xy)$. Show that I is maximal.

(b) Show that the localization A_I is isomorphic to $\mathbb{C}[x]_{(x-1)}$.

Hint: argue that $x = x/1$ is invertible in A_I. Deduce that $y = y/1$ is zero in A_I.

Note: the geometric picture is that the equation $x = 1$ defines a single point in the variety $xy = 0$, and this point is smooth.

Solution to Problem 1.

(a) V is given by the following two equations: $xz = 0$, $yz = 0$. Indeed, they hold whenever $z = 0$, or $y = 0$ and $x = 0$.

(b) Answer: $I = (xz, yz)$. It is clear that $(xz, yz) \subset I$ because both functions xz, yz vanish on V. To prove the converse, observe that $I = (x) \cap (y, z)$, because the ideal (x) describes all functions vanishing on $\{x = 0\}$, and the ideal (y, z) describes all functions vanishing on $\{y = z = 0\}$.

Suppose $f \in (x) \cap (y, z) = I$. Since $f \in (x)$, we have $f = xg$ for a polynomial g. Next, it is clear that (y, z) is prime so $f = xg \in (y, z)$ implies $g \in (y, z)$. This means $f \in (xy, xz)$.

Note: an alternative proof that $I \subset (xz, yz)$. (Essentially equivalent to the previous one, but written in more elementary terms.) Suppose $f(x, y, z) = \sum a_{ijk} x^i y^j z^k$ is a polynomial vanishes identically on V. Here a_{ijk}, $i, j, k \geq 0$ are its coefficients.

Since f vanishes identically on $\{z = 0\}$, the polynomial $f(x, y, 0)$ is identically zero. This means that $a_{ij0} = 0$.

Since f vanishes identically on $\{x = y = 0\}$, the polynomial $f(0, 0, z)$ is identically zero. This means that $a_{00k} = 0$.

Combining this together, f is a sum of monomials of the form $x^i y^j z^k$ where:

$k > 0$, and

either $i > 0$ or $j > 0$.

This means that $x^i y^j z^k$ is divisible either by xz or by yz, respectively. In both cases, it implies that $x^i y^j z^k \subset (xz, yz)$.

Solution to Problem 2.

(a) There exists no ideal J such that $I \subset J \subset A$, $J \neq I$, $J \neq A$.

(b) Because $\mathbb{C}[x]$ is a principal ideal domain, every ideal $I \subset \mathbb{C}[x]$ is generated by one element: $I = (f)$. Over $\mathbb{C}$, f decomposes as a product of linear functions $f = (x - a_1) \dots (x - a_n)$. If $n > 1$, we have $(f) \subset (x - a_1) \subset A$ where both inclusions are strict, so (f) is not maximal.

If $f = x - a$, then the map $\mathbb{C}[x]/(f) \to \mathbb{C}$, given by $g \mapsto g(a)$, is an isomorphism. Indeed, it's clearly surjective, and if $g \in \mathbb{C}[x]$ is in the kernel, then g is divisible by $x - a$ so g is actually zero in $A/(f)$. We have shown that $\mathbb{C}[x]/(f)$ is a field, so (f) is maximal.

Solution to Problem 3.

The points p_i lie on the conic $\{xy = 0\}$ which is singular. By uniqueness, there is no other conic passing through these points, in particular, no smooth conic.

The points q_i lie on the smooth conic $\{x^2 + y^2 + z^2 = 0\}$. By uniqueness, there is no other conic passing through these points, in particular, no singular conic.

Projective transformations take conics to conics, and singular/smooth conics to singular/smooth ones. The statement clearly follows.

Solution to Problem 4.

(a)

$$\begin{cases} zy = x^2 \\ z^2y = x^3. \end{cases}$$

(b) In the affine coordinates using the chart $z = 1$, the solutions are $(0, 0)$ and $(1, 1)$. In the complement $z = 0$ to the affine chart, there is one more solution: $z = x = 0$. The complete answer is 3 solutions: $[0 : 0 : 1]$, $[1 : 1 : 1]$, $[0 : 1 : 0]$.

Solution to Problem 5.

(a) We want to prove that A/I is a field. Indeed,

$$A/I \cong \frac{\mathbb{C}[x, y]}{(xy, x - 1)} \cong \frac{\mathbb{C}[y]}{(y)} \cong \mathbb{C}.$$

(b) Since $x \notin I$, $1/x \in A_I$ by definition, so x is invertible. Now $xy = 0$ implies that $x^{-1}xy = y = 0$ in A_I. So we can add the relation $y = 0$ before localizing and write

$$A_I \cong \left(\frac{\mathbb{C}[x, y]}{(xy, y)}\right)_{(x-1)} \cong \mathbb{C}[x]_{(x-1)}.$$