Math 143 Elementary Algebraic Geometry, Fall 2018. Instructor: Dmitry Tonkonog

Practice Midterm

The actual midterm will be 80 minutes long. No notes, books or calculators are allowed.

**Problem 1.** Consider $\mathbb{C}^3$ with coordinates $x, y, z \in \mathbb{C}$. Let $V \subset \mathbb{C}^3$ be the union of the plane $\{ z = 0 \}$ with the line $\{ x = y = 0 \}$.

(a) Show that $V$ is an affine variety.

(b) Let $I \subset \mathbb{C}[x, y, z]$ be the ideal consisting of all polynomials $f(x, y, z)$ such that $f(p) = 0$ for all $p \in V$. Find a finite set of polynomials generating $I$ (with a proof).

**Problem 2.**

(a) What does it mean for an ideal $I \subset A$ to be maximal? Write the definition.

(b) Prove that every maximal ideal in $\mathbb{C}[x]$ has the form $(x - a)$ for some $a \in \mathbb{C}$.

**Problem 3.** Consider $\mathbb{C}P^2$ with homogeneous coordinates $[x : y : z]$. Consider the following 5 points in $\mathbb{C}P^2$:

$p_1 = [0 : 0 : 1],
\quad p_2 = [1 : 0 : 1],
\quad p_3 = [2 : 0 : 1],
\quad p_4 = [0 : 1 : 1],
\quad p_5 = [0 : 2 : 1].$

Consider 5 other points $q_1, \ldots, q_5 \in \mathbb{C}P^2$ which are pairwise distinct and all belong to the set $\{ x^2 + y^2 + z^2 = 0 \} \subset \mathbb{C}P^2$.

Does there exist a projective transformation $\phi: \mathbb{C}P^2 \to \mathbb{C}P^2$ such that $\phi(p_i) = q_i$, $i = 1, \ldots, 5$?

*Hint: find a conic passing through $p_1, \ldots, p_5$. Is it smooth?*

**Problem 4.** Consider the following system of equations in $\mathbb{C}^2$:

\[
\begin{cases}
    y = x^2 \\
    y = x^3.
\end{cases}
\]

(a) Rewrite it in projective coordinates $[x : y : z]$ for $\mathbb{C}P^2$.

(b) Find all solutions to this system of equations in $\mathbb{C}P^2$.

**Problem 5.**

(a) Consider the ideal generated by $I = (x - 1)$ in the ring $A = \mathbb{C}[x, y]/(xy)$. Show that $I$ is maximal.

(b) Show that the localization $A_I$ is isomorphic to $\mathbb{C}[x]_{(x-1)}$.

*Hint: argue that $x = x/1$ is invertible in $A_I$. Deduce that $y = y/1$ is zero in $A_I$.*

*Note: the geometric picture is that the equation $x = 1$ defines a single point in the variety $xy = 0$, and this point is smooth.*
Solution to Problem 1.

(a) $V$ is given by the following two equations: $xz = 0$, $yz = 0$. Indeed, they hold whenever $z = 0$, or $y = 0$ and $x = 0$.

(b) Answer: $I = (xz, yz)$. It is clear that $(xz, yz) \subset I$ because both functions $xz, yz$ vanish on $V$. To prove the converse, observe that $I = (x) \cap (y, z)$, because the ideal $(x)$ describes all functions vanishing on $\{x = 0\}$, and the ideal $(y, z)$ describes all functions vanishing on $\{y = z = 0\}$.

Suppose $f \in (x) \cap (y, z) = I$. Since $f \in (x)$, we have $f = xg$ for a polynomial $g$. Next, it is clear that $(y, z)$ is prime so $f = xg \in (y, z)$ implies $g \in (y, z)$. This means $f \in (xy, xz)$.

Note: an alternative proof that $I \subset (xz, yz)$. (Essentially equivalent to the previous one, but written in more elementary terms.) Suppose $f(x, y, z) = \sum a_{ijk} x^i y^j z^k$ is a polynomial vanishes identically on $V$. Here $a_{ijk}, i, j, k \geq 0$ are its coefficients.

Since $f$ vanishes identically on $\{z = 0\}$, the polynomial $f(x, y, 0)$ is identically zero. This means that $a_{ij0} = 0$.

Since $f$ vanishes identically on $\{x = y = 0\}$, the polynomial $f(0, 0, z)$ is identically zero. This means that $a_{00k} = 0$.

Combining this together, $f$ is a sum of monomials of the form $x^i y^j z^k$ where:

- $k > 0$, and
- either $i > 0$ or $j > 0$.

This means that $x^i y^j z^k$ is divisible either by $xz$ or by $yz$, respectively. In both cases, it implies that $x^i y^j z^k \subset (xz, yz)$.

Solution to Problem 2.

(a) There exists no ideal $J$ such that $I \subset J \subset A$, $J \neq I$, $J \neq A$.

(b) Because $\mathbb{C}[x]$ is a principal ideal domain, every ideal $I \subset \mathbb{C}[x]$ is generated by one element: $I = (f)$. Over $\mathbb{C}$, $f$ decomposes as a product of linear functions $f = (x - a_1) \ldots (x - a_n)$. If $n > 1$, we have $(f) \subset (x - a_1) \subset A$ where both inclusions are strict, so $(f)$ is not maximal.

If $f = x - a$, then the map $\mathbb{C}[x]/(f) \rightarrow \mathbb{C}$, given by $g \mapsto g(a)$, is an isomorphism. Indeed, it’s clearly surjective, and if $g \in \mathbb{C}[x]$ is in the kernel, then $g$ is divisible by $x - a$ so $g$ is actually zero in $A/(f)$. We have shown that $\mathbb{C}[x]/(f)$ is a field, so $(f)$ is maximal.

Solution to Problem 3.

The points $p_i$ lie on the conic $\{xy = 0\}$ which is singular. By uniqueness, there is no other conic passing through these points, in particular, no smooth conic.

The points $q_i$ lie on the smooth conic $\{x^2 + y^2 + z^2 = 0\}$. By uniqueness, there is no other conic passing through these points, in particular, no singular conic.

Projective transformations take conics to conics, and singular/smooth conics to singular/smooth ones. The statement clearly follows.

Solution to Problem 4.
(a) 
\[
\begin{aligned}
zy &= x^2 \\
\quad z^2 y &= x^3.
\end{aligned}
\]

(b) In the affine coordinates using the chart \( z = 1 \), the solutions are \((0, 0)\) and \((1, 1)\). In the complement \( z = 0 \) to the affine chart, there is one more solution: \( z = x = 0 \). The complete answer is 3 solutions: \([0 : 0 : 1], [1 : 1 : 1], [0 : 1 : 0]\).

*Solution to Problem 5.*

(a) We want to prove that \( A/I \) is a field. Indeed,

\[
A/I \cong \frac{\mathbb{C}[x, y]}{(xy, x - 1)} \cong \frac{\mathbb{C}[y]}{(y)} \cong \mathbb{C}.
\]

(b) Since \( x \notin I \), \( 1/x \in A_I \) by definition, so \( x \) is invertible. Now \( xy = 0 \) implies that \( x^{-1}xy = y = 0 \) in \( A_I \). So we can add the relation \( y = 0 \) before localizing and write

\[
A_I \cong \left( \frac{\mathbb{C}[x, y]}{(xy, y)} \right)_{(x-1)} \cong \mathbb{C}[x]_{(x-1)}.
\]